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This Volume of

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is being dedicated to

Paṇini

Paṇini was a Sanskrit grammarian, logician, philologist, and revered scholar in ancient India, variously dated between the 7th and 4th century BCE. Panini was an ancient Vedic Indian Mathematician and Father of linguist. After all he is widely regarded as the father of Sanskrit grammar. His magnum opus, the Aṣṭadhyaya is a comprehensive and systematic treatise on Sanskrit morphology, phonology, and syntax. Panini's work had a profound impact on the development of Indian linguistics and literature. His rules and principles were adopted by later grammarians and scholars, and his influence can be seen in the works of many classical Indian authors.

In addition to his contributions to linguistics, Panini also made significant contributions to the development of mathematics. He introduced the concept of zero and developed a system of mathematical notation that is still used in India today. He also made contributions to the study of astronomy and physics. Panini's work is considered to be one of the most important contributions to Indian culture and scholarship. He is remembered as a brilliant and innovative thinker whose work has had a lasting impact on the world.

FOREWORD

The present volume of ***INSPIRE*** contains the various research papers of Faculty and Research Scholars of Department of Mathematics, INSTITUTE FOR EXCELLENCE IN HIGHER EDUCATION, BHOPAL (M. P.).

For me it is the realization of a dream which some of us have been nurturing for long and has now taken a concrete shape through the frantic efforts and good wishes of our dedicated band of research workers in our country, in the important area of mathematics.

The editor deserves to be congratulated for this very successful venture. The subject matter has been nicely and systematically presented and is expected to be of use to the workers.

(Dr. Pragyesh Kumar Agarwal)
Director & Patron
IEHE, Bhopal (M. P.)

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STUDY OF VARIATION IN Dst INDEX AND OCCUEENCE OF SEISMIC EVENT

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ABSTRACT

This review summarizes the observational and statistical literature on whether variations in the geomagnetic Dst index (a widely used measure of ring-current-driven geomagnetic storms) are associated with changes in earthquake occurrence. I summarize background physical ideas, the main empirical approaches used (superposed epoch analysis, time-lag/binomial tests, spectral/wavelet approaches), highlight representative positive and null findings, discuss methodological pitfalls and biases, and outline a research agenda to clarify whether a real, reproducible link exists and, if so, by what mechanism.

1. INTRODUCTION:

Interest in solar–terrestrial influences on seismicity has waxed and waned for decades. The Dst (Disturbance storm time) index is a one-number daily (and hourly) measure of the strength of the equatorial ring current, commonly used to identify geomagnetic storms. Several recent statistical studies have reported apparent clustering of geomagnetic storms near the times of large earthquakes; other studies, however, find no robust association once catalogue biases and statistical pitfalls are controlled for. The result is a mixed literature that requires careful, method-aware synthesis.

The Dst index (hourly values; negative values indicate enhanced westward ring current and storm-time depressions of the equatorial magnetic field) is produced from a network of low-latitude magnetometers and widely used to quantify storm magnitude and timing. Geomagnetic storms are driven by solar wind structures (CMEs, high-speed streams) and are accompanied by large variations in ionospheric currents, magnetospheric electric fields, energetic particle precipitation, and induced ground-level magnetic-and-electric fields. These space-weather changes can (in principle) lead to electromagnetic induction in the crust and lithosphere, changes in pore-fluid pressures via electrokinetic coupling, or modulation of shallow geoelectric conditions — all suggested pathways by which external electromagnetic forcing could, hypothetically, alter near-failure stress conditions on faults. Mechanistic plausibility has been proposed, but remains highly speculative and quantitatively unproven.

Empirical approaches used in the literature

Researchers have used a handful of common approaches:

1. **Superposed epoch analysis (SEA)** — Examine average Dst (or storm counts) in windows before/after earthquakes and test whether pre-event or post-event means deviate from random expectation.

2. **Time-lag/binomial / shift-matching tests** — Shift storm/event series relative to each other to assess time-lagged association (e.g., increases at particular lags like ~27–28 days).
3. **Case studies** — Detailed investigations of individual large earthquakes and nearby ionospheric/geomagnetic disturbances.
4. **Spectral and wavelet analyses** — look for common periodicities or coherence between seismicity rates and geomagnetic indices.

Each method has strengths and vulnerabilities. SEA is intuitive but sensitive to temporal clustering in seismic catalogs (aftershock sequences), catalog completeness, and the choice of "isolated" versus clustered events. Time-lag searches risk multiple-testing problems (search many lags and you will find some apparently significant peaks by chance). Case studies can be suggestive but cannot establish generality.

2. RESULTS:

2.1 Studies reporting confident relations

A number of statistical analyses have reported increased geomagnetic-storm activity in the days prior to major earthquakes (e.g., $M \geq 7.0$ global events during 1957–2020), using SEA and significance testing to argue that more geomagnetic storms occur before large earthquakes than after them; some studies report strongest signals ~7–10 days pre-event for "isolated" large events.

More recent work analysing nearly a century of Dst/Kp data has reported increases in earthquake counts following intense geomagnetic storms, and specific analyses have suggested an increased earthquake probability at particular time span.

Conversely, several analyses find little or no robust relationship once careful controls are applied. Time-series and statistical reanalyses conclude that apparent correlations can be produced by catalog non-stationarity, aftershock contamination, multiple testing, or arguably insufficient control of confounders. These works caution that the bulk of seismic energy release is governed by tectonics and internal stress evolution, and that external solar–geomagnetic forcing (if real) would be a small modulating term, lags (for example ~27–28 days after intense storms) using shift-matching correlation methods. These studies argue for a non-random association at particular timescales.

2.2 Studies reporting null or weak results: Number of studies also report a weak connection in between variation in Dst index and seismic occurrence**.**

3. CHALLENGES:

1. **Catalog completeness and delustering.** Aftershock sequences produce strong temporal clustering; failure to decluster or to restrict to "isolated" mainshocks can create spurious associations. Several positive studies attempt to use isolated-event subsets, but different declustering algorithms change results.

2. **Multiple testing and data dredging.** Searching many windows, lags, magnitudes, and indices without correcting for the number of trials inflates false positives. The reported ~27–28 day lag peaks are an example where multiple-lag searching may amplify chance findings.
3. **Physical mechanism and effect size.** Proposed mechanisms (electrokinetic pumping, piezoelectric modulation, stress changes from induced currents) are often qualitative; quantitative estimates of induced stress perturbations versus tectonic stress drops are typically orders of magnitude smaller. That mismatch challenges causal interpretation even when statistical associations appear.
4. **Selection bias and hindsight.** Choosing well-known large earthquakes and then searching nearby records for anomalies risks confirmation bias. Prospective forecasting tests (pre-registered hypotheses tested on out-of-sample data) are rare.
5. **Index choice and data sources.** Different geomagnetic indices (Dst, Kp, AE, local magnetometer components, TEC) measure different aspects of magnetospheric/ionospheric disturbance; the association may be index-dependent. Studies that combine multiple indices increase opportunities for spurious cross-correlations unless carefully corrected.

4. RECCOMENDATIONS FOR FUTURE RESEARCH:

1. **Pre-registration and out-of-sample testing.** Define hypotheses and analysis pipelines before examining data; evaluate on withheld time periods or future data to avoid overfitting.
2. **Consistent declustering and sensitivity checks.** Report results for multiple declustering algorithms and for both global and regionally confined catalogs.
3. **Multi-index approach with corrections.** Limit the number of indices/lags tested, apply multiple-comparison corrections, and be transparent about all trials.
4. **Mechanistic modeling.** Build quantitative models that estimate induced stress or pore-pressure changes from realistic geomagnetic/ionospheric perturbations, then compare those to typical fault critical stress thresholds.
5. **Regional studies.** If a true effect exists it may be conditional on lithology, crustal conductivity, or tectonic state; focused regional analyses (with high local magnetometer coverage and dense seismic catalogs) could be more sensitive than global aggregation.

5. CONCLUSION:

There is active, modern interest in whether geomagnetic storms (as measured by Dst and related indices) influence earthquake occurrence. Some well-executed statistical studies report signals suggesting non-random temporal association, but others find no robust link once methodological issues are controlled.

The balance of evidence currently points to inconclusive results: intriguing statistical hints exist but do not yet demonstrate a strong, reproducible causal effect. Resolving the question will require tightly pre-registered hypothesis tests, careful control of seismic catalog biases, quantitative mechanistic modelling that shows a plausible effect size, and replication on independent datasets.

REFERENCES:

1. A Statistical Study of the Correlation between Geomagnetic Storms and M ≥ 7.0 Global Earthquakes during 1957–2020. *Scientific Reports / PubMed/PMC*. (Study using SEA and Z-test reporting higher probability of storms before large earthquakes).
2. On the possible relationship between magnetic storms and earthquakes in certain tectonic conditions (Sakhalin Island example). *Izvestiya / regional study*. (Discusses regional analyses and mixed results).
3. "Correlation Between Intense Geomagnetic Storms and Global Strong Earthquakes" (AGU Geophysical Research Letters / 2024). (Reports increases in earthquakes at specific lags, e.g., ~27–28 days after intense storms). (
4. Is the Apparent Correlation between Solar-Geomagnetic Activity and Occurrence of Powerful Earthquakes a Casual Artifact? (Methodological reanalysis arguing against strong correlations). *Preprint/pdf on semanticscholar; peer-reviewed discussions cite this analysis.*
5. Statistical Analysis of the Correlation between Geomagnetic Storm Intensity and Solar Wind Parameters (Remote Sensing, 2024). (Provides context on Dst storms, data sources, and storm identification used by many seismic-geomagnetic studies).
6. Studies of ionospheric/geomagnetic precursors and seismic activity (case studies and ionospheric TEC work). *Various papers (e.g., studies using TEC/VLF and local magnetometer arrays) that examine ionospheric disturbances near seismic events.*

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REVIEW ARTICLE ON BARDEEN - COOPER AND SCHRIEFFER THEORY OF SUPERCONDUCTING PHENOMENA

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ABSTRACT

The aim of, this review article to study the BCS theory with superconducting experimental aspects for different materials. The BCS theory is solely based on the electron-phonon-electron interaction to explain the conventional superconductivity in materials. It became a widely used theory to predict the nature of materials. However, time to time it failed to explain the continuous growing number of superconducting phenomena classified as 'unconventional'. It could not account for many unknown variables like coulombic pseudo potential and magnetic spin fluctuation. Hence a need to revisit the for understanding and more importantly the application of BCS theory is essential.

1. Introduction:

Many theories which we learn today have been there for quite some time now, but our textbooks forget to update them with time and further students grow up to become researchers but do not bother to ask the most fundamental question: does these theories actually work? Hence a time-to-time revisit of such theories should be done to understand their current relevance to the field, which is what we achieve in this review paper.

One such theory is the BCS theory. The Bardeen-Cooper-Schrieffer (BCS) theory revolutionized our understanding of superconductivity by providing the first microscopic explanation for this phenomenon. Its success in explaining conventional low-temperature superconductors, such as lead (Pb) and mercury (Hg), earned widespread acceptance. [1] However, the discovery of high-temperature superconductivity in 1986 and other future developments have raised a question on whether BCS theory can fully account for all forms of superconductivity or not? [2]

In this review, we reexamine the foundations of BCS theory, highlighting areas where it has successfully predicted superconducting behavior, as well as the challenges posed by materials and phenomena that lie beyond its scope. We explore alternative theories and experimental findings that suggest the need for a revised or extended framework for understanding superconductivity.

2. BCS: why we accept it widely:

It is one of the few theories in the history of Physics which won the Nobel prize and for a good reason. It offered a different perspective and almost approximate theoretical results for the superconductors present then. A theory worth presenting a Nobel prize for sure overachieved many milestones.

Some factors which undoubtedly contributed in proving the theory right are as followed.

2.1. The Part which works

The BCS theory is based on the concept of Cooper pairs, where electrons with opposite momentum and spins form bound states with phonons in between. These pairs condense into a macroscopic quantum state, forming electron phonon electron interactions, resulting in zero electrical resistance [3] BCS theory also explains the energy gap, which is a consequence of the pairing mechanism, providing a barrier to electron scattering and thereby eliminating electrical resistance below the critical temperature Tc.[4]

It explained zero resistance perfectly. As the Cooper pairs create an energy gap at the Fermi surface, which leads to zero electrical resistance. The electrons together cancel out each other's spin and act like bosons, which share the same state, allowing them to coordinate their movement and reduce resistance to zero. This formation of Cooper pairs also prevents scattering events that would normally cause resistance in conventional metals.[3]

The theory undoubtedly was able to explain the concepts of Cooper pairs, of macroscopic phase coherence, and the existence of an energy gap. The electron phonon electron interaction leading to the formation of Cooper pairs explained how the smooth movement of the pair takes place in a material. The macroscopic phase coherence refers to how the wave function of the Cooper pairs is locked together, maintaining a fixed phase relationship throughout the material. This helps in superconducting current to flow. This also leads to a coherent state which further makes a energy band gap in the electronic density of states, this means to break the Cooper pairs down we need a certain amount of energy. [5]

These key elements of the theory have successfully explained and even predicted perplexing experimental observations, such as the nuclear magnetic resonance (NMR) relaxation rate [6] and Josephson tunneling [7]. However, despite their wide acceptance, it can be seen that several other aspects of BCS theory, particularly those involving the electron–phonon interaction as the driving mechanism, are incorrect and warrant further scrutiny.

2.2 The Endurance of BCS Theory

Many physicists argue that the long-standing acceptance of BCS theory, spanning more than 50 years, is evidence of its correctness. However, it is very much possible that the flaws in theories can be detected anytime even after its over and over use. All it takes is one experimental result and we have many in the case of BCS theory. [8]

2.3 The support of conventional superconductors

The most widely cited evidence supports the idea that BCS electron phonon theory explains conventional superconductors Congress from the tunneling experiments. In these experiments, small "wiggles" or variations in the tunneling conductance (the current passing through a barrier between a normal metal, an insulator, and a superconductor) match the Maximas and Minima in the phonon density of states (the distribution of vibrational energy in the crystal lattice) measured through neutron scattering experiments. This correlation, proving the electron phonon interaction, has been observed in several materials, particularly lead (Pb). [9-11]

The interpretation of tunneling results is cast in terms of the spectral function α^2F(ω), where F(ω) is the phonon spectral function determined from neutron scattering experiments. What is not emphasized is that α is itself often a strong function of ω that is not directly accessible to experiment. [12]

3. Challenges to BCS Theory

The challenges of BCS theory started to occur in the early 1980s when new materials were rapidly being discovered. Our beloved BCS theory however was unable to explain the superconductivity in many of these modern materials. New terms emerged and challenges continued to grow. Results which the complex formulism of BCS theory could not explain.

3.1. High-Temperature Superconductors

The discovery of high-temperature superconductivity (HTSC) in copper-oxide (cuprate) materials by Bednorz and Müller in 1986 posed the first major challenge to the BCS framework. These materials exhibit critical temperatures far higher than those predicted by the electron-phonon coupling mechanism central to BCS theory. Additionally, the pairing symmetry in HTSCs is d-wave rather than the conventional s-wave symmetry assumed in BCS theory, indicating that the pairing mechanism may be fundamentally different.[13]

3.1.1. Breakdown of the Phonon Mechanism

One of the key tenets of BCS theory is that electron pairing is mediated by phonons—vibrations of the crystal lattice. However, in high-temperature superconductors, the phonon-based mechanism appears insufficient to explain the high Tc . It is difficult to understand how an electron phonon electron interaction can overcome strong coulombic repulsion between the two electrons . This has led researchers to propose alternative mechanisms, such as pairing mediated by spin fluctuations or other electronic interactions.[14]

3.1.2. Pseudogap Phenomenon

Another puzzling feature in high-temperature superconductors is the pseudogap phase, a state in which a partial energy gap forms above the superconducting transition temperature. This phase is not easily reconcilable with BCS theory, which predicts a full energy gap only below Tc. The origin of the pseudogap remains a phenomenon which we know very little about, however it can definitely be experimentally observed in materials. It seems to be more complex than what BCS theory can explain. [15]

3.2. Iron-Based Superconductors

In 2008, the discovery of iron-based superconductors (FeSCs) provided another system that challenges the BCS framework. While FeSCs display superconductivity at relatively high temperatures, their electronic structure and pairing mechanisms differ significantly from those of both conventional superconductors and cuprates. Specifically, FeSCs exhibit multiple Fermi surfaces and unconventional pairing symmetries, further straining the applicability of the BCS formalism. As Unlike traditional superconductors, IBS do not contain copper or other common conductive elements. Instead, they rely on iron and oxygen atoms arranged in a specific crystal structure to exhibit superconductivity. [16-18]

3.3. Exotic Superconductors

In addition to high-temperature and iron-based superconductors, several exotic systems have been discovered that do not conform to the predictions of BCS theory. These include:

1. Heavy fermion superconductors: In materials like UPt_3 and $CeCoIn_5$, strong electronic
   correlations dominate, and the superconducting pairing mechanism is thought to involve magnetic fluctuations rather than phonons.
2. Topological superconductors: These systems exhibit non-trivial topological order and host exotic quasiparticles, such as Majorana fermions. BCS theory, rooted in the assumption of conventional pairing, is not equipped to explain the unique properties of these materials.
3. Superconductivity in doped semimetals and twisted bilayer graphene: These recently discovered superconductors exhibit unconventional pairing mechanisms that do not easily fit within the BCS framework.[19-20]

3.4 Inability to explain the Meissner effect

The Meissner effect is one of the most fundamental characteristics of superconductors. When a superconductor is cooled in the presence of a static magnetic field, an electric current spontaneously develops near its surface, effectively expelling the magnetic field from the interior of the material . However, conventional superconductivity theory fails to address two key questions:

- How do the electrons near the surface of the superconductor acquire the necessary velocity to screen the magnetic field inside?
- How is angular momentum conserved during this process?

These are fundamental questions that relate directly to the core nature of superconductivity.[8]

In response to the first question, a conventional theorist might argue that because the final state, with supercurrent flowing, has a lower free energy than the initial state, the system will naturally transition to this state. However, the supercurrent is a macroscopic phenomenon, and there should be a clearly identifiable macroscopic force that causes the electrons near the surface to move in the same direction to generate the necessary current.

A typical response would be that the force driving this motion is $-dF/dx$ (the change in free energy over distance), and no further explanation is needed. Yet, this reasoning is flawed.[22]

Contrary to this explanation, Faraday's law predicts the presence of an induced electric field, which exerts a force on the charge carriers in the opposite direction of what is needed to generate the Meissner current . For the supercurrent to form, the superconductor must overcome this opposing force with another force acting in the correct direction on the superfluid carriers. The $-dF/dx$ term, which suggests a force along the azimuthal direction (needed to create the Meissner current), does not correspond to a real, physical force. The only relevant forces in this context (excluding gravitational and nuclear forces) are the Lorentz electromagnetic force and quantum pressure, which refers to the natural tendency of quantum particles to reduce their kinetic energy by expanding their wavefunctions . However, neither of these forces plays a role in the Meissner effect according to conventional superconductivity theory.[23]

Addressing the second question, regarding angular momentum conservation, is even more challenging within the conventional framework. In the final state, the supercurrent carries mechanical angular momentum, while the total angular momentum in the normal state is zero, creating a situation of "missing angular momentum". A common explanation is that this angular momentum is transferred to the ionic lattice, but the conventional theory does not provide a clear mechanism for how this transfer would occur. If the electrons were to transfer angular momentum to the lattice through scattering with impurities or phonons, this process should be clearly described, as the ions involved are essentially classical objects. Yet, no such explanation has been provided, and it has been argued that it may be impossible to describe this process within the framework of conventional superconductivity theory .[24-28]

4. Alternative Theoretical Approaches

Given the limitations of BCS theory in explaining non-conventional superconductors, several alternative and extended theoretical models have been proposed.

4.1. Eliashberg Theory

Eliashberg theory extends the BCS formalism to account for strong electron-phonon interactions and retardation effects. While this approach has been successful in explaining certain strong-coupling superconductors, it still relies on phonon-mediated pairing, limiting its applicability to materials where the electron-phonon interaction is dominant.[29]

4.2. Spin-Fluctuation Mediated Pairing

In both cuprate and iron-based superconductors, spin fluctuations are believed to play a crucial role in mediating the pairing interaction. This theory posits that the exchange of virtual spin excitations, rather than phonons, leads to electron pairing. This mechanism naturally explains the d-wave pairing symmetry observed in cuprates and the unconventional pairing in FeSCs.[30]

4.3. Quantum Criticality and Superconductivity

Many unconventional superconductors, including heavy fermion and iron-based systems, exhibit superconductivity in the vicinity of a quantum critical point (QCP)—a point at which a continuous phase transition occurs at absolute zero. Quantum fluctuations near the QCP may provide the pairing mechanism in these materials, suggesting that superconductivity in these systems is fundamentally tied to quantum critical behavior, a phenomenon outside the scope of BCS theory.[31]

5. Experimental Evidence Challenging BCS Predictions

Several experimental findings provide evidence for superconducting behavior that deviates from BCS predictions:

5.1. Nodal Structures and Gap Anisotropy

In many unconventional superconductors, such as cuprates and heavy fermion systems, the superconducting gap is highly anisotropic, with nodes (zero-gap points) in certain directions. This stands in stark contrast to the isotropic gap predicted by BCS theory for s-wave superconductors.[32]

5.2. Unconventional Symmetry

Experiments on materials like strontium ruthenate (Sr_2RuO_4) and uranium-based superconductors suggest that the pairing symmetry in these systems may be odd-parity or spin-triplet, which is incompatible with the even-parity spin-singlet pairing assumed in BCS theory.[33]

6. Conclusion: Is It Time to Move Beyond BCS?

While this paper has focused primarily on the BCS theory, the broader implications extend across various domains of contemporary science. The same forces that contribute to the persistence of BCS theory may also be sustaining other flawed scientific frameworks today. As knowledge grows and becomes more specialized, incoming students increasingly depend on "gatekeepers" – professors, mentors, and established scientists – to guide their entry into the scientific community. These gatekeepers often have vested interests in maintaining the status quo. A young scientist with revolutionary ideas that challenge established norms may face strong discouragement, and even risk being denied opportunities in the field if they persist. By the time scientists reach an established position, they are typically conditioned to accept the dominant truths.

In the case of BCS theory, it would be beneficial for journal editors to be more open to papers that critically evaluate or challenge its validity, while recognizing that some referees may have vested interests in rejecting such work. Allowing these critical papers to be published in mainstream journals would encourage both younger scientists and seasoned experts, especially those who are starting to have doubts in light of recent experimental findings, to explore alternatives to conventional BCS theory.

Funding agencies should allocate at least a small portion of resources to support experimental and theoretical studies that question BCS theory. Likewise, conference and workshop organizers should invite speakers whose research critically examines the theory, rather than avoiding these discussions.

Despite being more than fifty years old, BCS theory has never successfully predicted a high-temperature superconductor and provides no useful guidance for the discovery of new superconducting materials. Furthermore, it has failed to explain the superconductivity of ten families of compounds discovered over the last three decades, as well as fundamental phenomena such as the Meissner effect and the behavior of rotating superconductors.

It is time to seriously consider that the stagnation in our understanding of high-temperature cuprates and other unconventional superconductors may stem from our failure to fully understand conventional superconductors. The possibility must be acknowledged that BCS theory is fundamentally flawed, and just as other long-standing scientific theories have been overturned in the past, it may soon be time for BCS theory to undergo a significant overhaul.

References:

1. Bardeen, J., Cooper, L. N., & Schrieffer, J. R. (1957). "Theory of Superconductivity." Physical Review, 108(5), 1175–1204. DOI: 10.1103/PhysRev.108.1175.
2. Bednorz, J. G.; Müller, K. A. (June 1986). "Possible high Tc superconductivity in the Ba−La−Cu−O system". Zeitschrift für Physik B: Condensed Matter. doi:10.1007/BF01303701. S2CID 118314311.
3. Bardeen, J.; Cooper, L. N.; Schrieffer, J. R. (April 1957). "Microscopic Theory of Superconductivity". Physical Review. 106 (1): 162–164. Bibcode:1957PhRv..106..162B
4. “University Physics Volume 3 Textbook “ by Jeff Sanny, Samuel J. Ling, and William Moebs
5. de Gennes, P. G. “Superconductivity of Metals and Alloys”. Westview Press, 1999
6. Carbotte J C 1990 Properties of boson-exchange superconductors Rev. Mod. Phys. 62 1027–157
7. Hirsch J E 2009 BCS theory of superconductivity: the world’s largest Madoff scheme? arXiv: 0901.4099v1[physics.gen-ph]
8. J E Hirsch, (2009),BCS theory of superconductivity: it is time to question its validity ,0031-8949/80/03/035702
9. McMillan W L and Rowell J M 1969 Tunneling and strong-coupling superconductivity Superconductivity vol I ed R D Parks (New York: Dekker) pp 561–613
10. Scalapino D J, Schrieffer J R and Wilkins J W 1966 Strong-coupling superconductivity I. Phys. Rev. 148 263–79.

11. Scalapino D J 1969 The electron–phonon interaction and strong-coupling superconductors Superconductivity ed R D Parks (New York: Dekker) pp 449–560
12. Weber W 1984 The phonons in high Tc A15 compounds Physica B+C 126 217–28
13. Timmer, John (May 2011). "25 years on, the search for higher-temp superconductors continues". Ars Technica. Archived from the original on 4 March 2012. Retrieved 2 March 2012.
14. Breakdown of Optical Phonons' Splitting in Two-Dimensional Materials by Thibault Sohier,OrcidMarco Gibertini,Matteo Calandra,Francesco Mauri,Nicola Marzari.
15. Timusk, Tom; Bryan Statt (1999). "The pseudogap in high-temperature superconductors: an experimental survey". Reports on Progress in Physics. 62 (1): 61–122. arXiv:cond-mat/9905219. Bibcode:1999RPPh...62...61T. doi:10.1088/0034-4885/62/1/002. S2CID 17302108.
16. Hosono, H.; Tanabe, K.; Takayama-Muromachi, E.; Kageyama, H.; Yamanaka, S.; Kumakura, H.; Nohara, M.; Hiramatsu, H.; Fujitsu, S. (2015). "Exploration of new superconductors and functional materials, and fabrication of superconducting tapes and wires of iron pnictides". Science and Technology of Advanced Materials. 16 (3): 033503. arXiv:1505.02240. Bibcode:2015STAdM..16c3503H. doi:10.1088/1468-6996/16/3/033503. PMC 5099821. PMID 27877784.
17. Kamihara, Yoichi; Hiramatsu, Hidenori; Hirano, Masahiro; Kawamura, Ryuto; Yanagi, Hiroshi; Kamiya, Toshio; Hosono, Hideo (2006). "Iron-Based Layered Superconductor: LaOFeP". J. Am. Chem. Soc. 128 (31): 10012–10013. doi:10.1021/ja063355c. PMID 16881620.
18. Kamihara, Yoichi; Watanabe, Takumi; Hirano, Masahiro; Hosono, Hideo (2008). "Iron-Based Layered Superconductor La[O1−xFx]FeAs (x = 0.05–0.12) with Tc = 26 K". Journal of the American Chemical Society. 130 (11): 3296–3297. doi:10.1021/ja800073m. PMID 18293989.
19. Bonn, D. Are high-temperature superconductors exotic?. Nature Phys 2, 159–168 (2006). https://doi.org/10.1038/nphys248
20. "Heavy-fermion systems" by Stewart, G. R. DOI 10.1103/RevModPhys.56.75
21. Meissner W and Ochsenfeld R 1933 Ein neuer Effekt bei Eintritt der Supraleitfhigkeit Naturwissenschaften 21 787–8
22. Hirsch J E 2009 Electromotive forces and the Meissner effect puzzle arXiv: 0908.4096v2 [cond-mat.supr-con]
23. Hirsch J E 2007 Do superconductors violate Lenz's law?
24. Body rotation under field cooling and theoretical implications Phys. Lett. A 366 615–9
25. [125] Hirsch J E 2009 Electromotive forces and the Meissner effect puzzle arXiv: 0908.4096v2 [cond-mat.supr-con].

26. [126] Hirsch J E 2008 Spin Meissner effect in superconductors and the origin of the Meissner effect Europhys. Lett. 81 67003
27. [127] Hirsch J E 2008 Electrodynamics of spin currents in superconductors Ann. Phys., Berlin 17 380–409
28. [128] Hirsch J E 2009 Charge expulsion, spin Meissner effect, and charge inhomogeneity in superconductors J. Supercond.Novel Magn. 22 131–9
29. F. Marsiglio 'Eliashberg Theory: a short review' arXiv:1911.05065 [cond-mat.supr-con]
30. T. Hattori, Y. Ihara, Y. Nakai, K. Ishida, Y. Tada, S. Fujimoto, N. Kawakami, E. Osaki, K. Deguchi, N. K. Sato, and I. Satoh, Phys. Rev. Lett. 108, 066403 (2012)
31. Jost, D., Peis, L., He, G. et al. Quantum critical fluctuations in an Fe-based superconductor. Commun Phys 5, 201 (2022). https://doi.org/10.1038/s42005-022-00981-5
32. Torsello, D., Piatti, E., Ummarino, G.A. et al. Nodal multigap superconductivity in the anisotropic iron-based compound RbCa2Fe4As4F2. npj Quantum Mater. 7, 10 (2022). https://doi.org/10.1038/s41535-021-00419-1
33. Zhang, Z., Wu, Z., Fang, C. et al. Topological superconductivity from unconventional band degeneracy with conventional pairing. Nat Commun 15, 7971 (2024). https://doi.org/10.1038/s41467-024-52156-1
34. Bednorz, J. G., & Müller, K. A. (1986). Possible High Tc Superconductivity in the Ba-La-Cu-O System. Zeitschrift für Physik B Condensed Matter, 64(2), 189-193.
35. Anderson, P. W. (2007). Is There Life After BCS? Reflections on the Future of Superconductivity. Journal of Physics and Chemistry of Solids, 67(1-3), 1-5.
36. Scalapino, D. J. (2012). A Common Thread: The Pairing Interaction for Unconventional Superconductors. Reviews of Modern Physics, 84(4), 1383-1417.
37. Stewart, G. R. (2011). Superconductivity in Iron-Based Superconductors. Reviews of Modern Physics, 83(4), 1589-1652.
38. Norman, M. R., & Pépin, C. (2003). The Electronic Nature of High-Temperature Cuprate Superconductors. Reports on Progress in Physics, 66(10), 1547-1610.

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FIXED POINT THEOREM FOR RATIONAL TYPE CONTRACTION IN S-METRIC SPACES

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ABSTRACT

The goal of this paper is to define rational contraction in the context of S-metric spaces and develop various fixed-point theorems in order to elaborate, generalize, and synthesize a number of previously published results. Finally, to illustrate the new theorem, an example is given.

KEYWORDS:S-metric space; rational contraction; fixed point.
MSC: Primary 47H10; Secondary 54H25

1. Introduction:

Fixed point theory is crucial in science and mathematics. This topic has drawn a lot of interest from academics in the last two decades due to its wide range of applications in disciplines such as nonlinear analysis, topology, and engineering difficulties. The Banach contraction principle [2] is the starting point for most generalizations of metric fixed point theorems. It's difficult to enumerate all of this principle's generalizations. The Banach fixed-point theorem [2] ensures the existence and uniqueness of fixed points of particular self-maps of metric spaces, as well as a constructive approach for discovering them. The S-metric space was introduced by Sedghi et al. [9]. It's a three-dimensional space called the S-metric space. The concept of A-metric space was established by Abbas et al. [1], which is a generalization of S-metric space. Jaggi [7], Das and Gupta [3] discovered the fixed-point theorem for rational contractive type conditions in metric space. The goal of this paper is to define rational contraction in the setting of S-metric spaces, as well as to create various fixed-point theorems to elaborate, generalize, and synthesize several previously published results. Finally, an example is given to demonstrate the new theorem.

2. Preliminaries

Some valuable information and ideas will be presented in this section. Metric spaces are very important in mathematics and applied sciences. So, some authors have tried to give generalizations of metric spaces in several ways. Sedghi et al. [8, 10] introduced the notion of a D^*-metric space as follows.

Definition 2.1 (see [8, 10]) Let X be a non-empty set. A D^*-metric on X is a function $D^*: X^3 \to [0, +\infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$;

(D*1). $D^*(x, y, z) \geq 0$,
(D*2). $D^*(x, y, z) = 0$if and only if $x = y = z$.

(D*3). $D^*(x, y, z) = D^*\{x, y, z\}$ (Symmetry in all three variables),

(D*4). $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

Then D^* is called a D^*-metric on X and (X, D^*) is called a D^*-metric space.

Definition 2.2 (see [9]) Let X be a nonempty set. A mapping $S: X^3 \to [0, +\infty)$ is called an S-metric on X if and only if for all $x, y, z, a \in X$, the following conditions hold:

(S1). $S(x, y, z) \geq 0$,

(S2). $S(x, y, z) = 0$ if and only if $x = y = z$,

(S3). $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$

The pair (X, S) is called an S-metric space.

The following is the intuitive geometric example for S-metric spaces.

Example 2.3 (see [9]) Let $X = R^2$ and d be the ordinary metric on X. Put

$$S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$$

for all $x, y, z \in X$, that is, S is the perimeter of the triangle given by x, y, z. Then S is an S-metric on X.

Example 2.4 Let $X = [1, +\infty)$. Define $S: X^3 \to [0, +\infty)$ by

$$S(y_1, y_2, y_3) = \sum_{i=1}^{3} \sum_{i<j} |y_i - y_j|$$

for all $y_i \in X, i = 1,2,3$.

Lemma 2.5 (see [9]) Let (X, S) be an S-metric space. Then for all $x, y \in X$,

$$S(x, x, y) = S(y, y, x).$$

Lemma 2.6 Let (X, S) be an S-metric space. Then for all $x, y, z \in X$,

$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z)$ and

$S(x, x, z) \leq 2S(x, x, y) + S(z, z, y)$.

Definition 2.7 (see [9]) Let X be an S-metric space.

(i). A sequence $\{y_n\}$ converges to y if and only if $S(y_n, y_n, y) = 0$. That is for each $\epsilon > 0$ there exists $n_0 \in N$ such that for all $n \geq n_0, S(y_n, y_n, y) < \epsilon$ and we denote this by

$$y_n = y.$$

(ii). A sequence $\{y_n\}$ is called a Cauchy if $S(y_n, y_n, y_m) = 0$. That is, for each $\epsilon > 0$ there exists $n_0 \in N$ such that for all $n, m \geq n_0, S(y_n, y_n, y_m) < \epsilon$.

(iii).X is called complete if every Cauchy sequence in X is a convergent.

From (see [9]), we have the following.

Example 2.8

(a). Let R be the real line. Then

$$S(x, y, z) = |x - z| + |y - z|$$

for all $x, y, z \in R$, is an S-metric on R. This S-metric is called the usual S-metric on R. Furthermore, the usual S-metric space R is complete.

(b). Let X be a non-empty set of R. Then

$$S(x, y, z) = |x - z| + |y - z|$$

for all $x, y, z \in X$, is an S-metric on X. If X is a closed subset of the usual metric space R, then the S-metric space X is complete.

Lemma 2.9 (see [9]) Let (X, S) be an S-metric space. If the sequence $\{y_n\}$ in X converges to y, then y is unique.
Lemma 2.10 (see [9]) Let (X, S) be an S-metric space. If

$$y_n = y \text{ and } z_n = z.$$

Then

$$S(y_n, y_n, z_n) = S(y, y, z).$$

Remark 2.11 It is easy to see that every D^*-metric is S-metric, but in general the converse is not true, see the following example.
Example 2.12 Let $X = R^n$ and $\|\,.\,\|$ a norm on X, then

$$S(x, y, z) = \|y + z - 2x\| + \|y - z\|$$

is S-metric on X, but it is not D^*-metric because it is not symmetric.
The following lemma shows that every metric space is an S-metric space.
Lemma 2.13 Let (X, d) be a metric space. Then we have
(1). $S_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$ is an S-metric on X.
(2). $x_n = x$ in (X, d) if and only if $x_n = x$ in (X, S_d).
(3). $\{x_n\}_{n=1}^{\infty}$ is Cauchy in (X, d) if and only if $\{x_n\}_{n=1}^{\infty}$ is Cauchy in (X, S_d).
(4). (X, d) is complete if and only if (X, S_d) is complete.

Example 2.14 Let $X = R$ and let

$$S(x, y, z) = |y + z - 2x| + |y - z|$$

for all $x, y, z \in X$. By ([9]), (X, S) is an S-metric space. Dung et al. [4] proved that there does not exist any metric d such that

$$S(x, y, z) = d(x, z) + d(y, z)$$

for all $x, y, z \in X$. Indeed, suppose to the contrary that there exists a metric d with

$$S(x, y, z) = d(x, z) + d(y, z)$$

for all $x, y, z \in X$. Then

$$d(x, z) = \frac{1}{2}S(x, x, z) = 2|x - z| \text{ and}$$

$$d(x, y) = \frac{1}{2}S(x, y, y) = 2|x - y|$$

for all $x, y, z \in X$. It is a contradiction.
In 2012, Sedghi et al. [9] asserted that an S-metric is a generalization of a G-metric, that is, every G-metric is an S-metric, see [9, Remarks 1.3] and [9, Remarks 2.2]. The Example 2.1 and Example 2.2 of Dung et al. [4] shows that this assertion is not correct. Moreover, the class of all S-metrics and the class of all G-metrics are distinct.
Definition 2.15 (see [11]) Let $(X, \preccurlyeq)$ be a partially ordered set and let $F: X \to X$ be a mapping. Then

1. elements $y, z \in X$ are comparable, if $y \preceq z$ or $z \preceq y$ holds;
2. a non empty set X is called well ordered set, if every two elements of it are comparable;
3. F is said to be monotone non-decreasing w.r.t. $\preceq$, if for all $y, z \in X$, $y \preceq z$ implies $Fy \preceq Fz$;

4. F is said to be monotone non-increasing w.r.t. $\preceq$, if for all $y, z \in X$, $y \preceq z$ implies $Fy \succeq Fz$.

3. Main Results

First, we introduce following definitions.

Definition 3.1 The triple $(X, S, \preccurlyeq)$ is called partially ordered S-metric spaces if $(X, \preccurlyeq)$ could be a partial ordered set and (X, S) be a S-metric space.

Definition 3.2 If X is complete S-metric, then $(X, S, \preccurlyeq)$ is called complete partially ordered metric space.

Definition 3.3 A partially ordered S-metric space $(X, S, \preccurlyeq)$ is called an ordered complete (OC), if for each convergent sequence $\{y_k\} \subset X$, the subsequent condition holds: either

- if $\{y_k\} \subset X$ is a non-increasing sequence such that $y_k \to y \in X$, then $y_k \preccurlyeq y$, for all $k \in N$, that is, $y = inf\ inf\ \{y_k\}$, or
- if $\{y_k\} \subset X$ is a non-decreasing sequence such that $y_k \to y$ implies that $y_k \preccurlyeq y$, for all $k \in N$, that is, $y = sup\ sup\ \{y_k\}$.

The following is our first main outcome.

Theorem 3.1 Let $(X, S, \preccurlyeq)$ be a complete partially ordred S-metric space. Suppose a self map F on X is continuous, non-decreasing and satisfies the contraction condition

$$S(Fy, Fy, Fz) \leq a\frac{S(y,y,Fy)S(z,z,Fz)}{S(y,y,z)} + b[S(y,y,Fy) + S(z,z,Fz)] + cS(y,y,z) \tag{3.1}$$

for any $y \neq z \in X$ with $y \preccurlyeq z$, where $a, b, c \in [0,1)$ with $0 \leq a + 2b + c < 1$. If $y_0 \preccurlyeq Fy_0$ for certain $y_0 \in X$, then F has a fixed point.

Proof Define a sequence, $y_{k+1} = Fy_k$ for $y_0 \in X$. If $y_{k_0+1} = y_{k_0}$ for certain $y_0 \in N$, then y_{k_0} is a fixed point F. Assume that $y_{k+1} \neq y_k$ for each k. But $y_0 \preccurlyeq Fy_0$ and F is non-decreasing as by induction we obtain that

$$y_0 \preccurlyeq y_1 \preccurlyeq y_2 \preccurlyeq \cdots \preccurlyeq y_k \preccurlyeq y_{k+1} \leq \cdots \tag{3.2}$$

By (3.1), we have

$$\begin{aligned} S(y_{k+1}, y_{k+1}, y_k) &= S(Fy_k, Fy_k, Fy_{k-1}) \\ &\leq a\frac{S(y_k,y_k,Fy_k)S(y_{k-1},y_{k-1},Fy_{k-1})}{S(y_k,y_k,y_{k-1})} \\ &\quad + b[S(y_k, y_k, Fy_k) + S(y_{k-1}, y_{k-1}, Fy_{k-1})] + cS(y_k, y_k, y_{k-1}) \\ &= a\frac{S(y_k,y_k,y_{k+1})S(y_{k-1},y_{k-1},y_k)}{S(y_k,y_k,y_{k-1})} \\ &\quad + b[S(y_k, y_k, y_{k+1}) + S(y_{k-1}, y_{k-1}, y_k)] + cS(y_k, y_k, y_{k-1}) \\ &= a\frac{S(y_{k+1},y_{k+1},y_k)S(y_k,y_k,y_{k-1})}{S(y_k,y_k,y_{k-1})} \\ &\quad + b[S(y_{k+1}, y_{k+1}, y_k) + S(y_k, y_k, y_{k-1})] + cS(y_k, y_k, y_{k-1}) \\ &= (a + b)S(y_{k+1}, y_{k+1}, y_k) + (b + c)S(y_k, y_k, y_{k-1}) \end{aligned}$$

which infer that

$$\begin{aligned} S(y_{k+1}, y_{k+1}, y_k) &\leq \left(\frac{b+c}{1-a-b}\right) S(y_k, y_k, y_{k-1}) \\ &\leq \left(\frac{b+c}{1-a-b}\right)^k S(y_1, y_1, y_0) \leq \cdots \end{aligned} \tag{3.3}$$

For $m, k \in N$ with $m > k$, by repeated use of (S3), we have

$$\begin{aligned}
S(y_k, y_k, y_m) &\leq 2S(y_k, y_k, y_{k+1}) + S(y_m, y_m, y_{k+1}) \\
&\leq 2S(y_k, y_k, y_{k+1}) + S(y_{k+1}, y_{k+1}, y_m) \\
&\leq 2S(y_k, y_k, y_{k+1}) + 2S(y_{k+1}, y_{k+1}, y_{k+2}) + S(y_m, y_m, y_{k+2}) \\
&\leq 2S(y_k, y_k, y_{k+1}) + 2S(y_{k+1}, y_{k+1}, y_{k+2}) + S(y_{k+2}, y_{k+2}, Sy_m) \\
&\leq 2S(y_k, y_k, y_{k+1}) + 2S(y_{k+1}, y_{k+1}, y_{k+2}) + 2S(y_{k+2}, y_{k+2}, y_{k+3}) \\
&\quad + S(y_m, y_m, y_{k+3}) \\
&\leq 2S(y_k, y_k, y_{k+1}) + 2S(y_{k+1}, y_{k+1}, y_{k+2}) + 2S(y_{k+2}, y_{k+2}, y_{k+3}) \\
&\quad + 2S(y_{k+3}, y_{k+3}, y_{k+4}) + \cdots + 2S(y_{m-2}, y_{m-2}, y_{m-1}) \\
&\quad + S(y_{m-1}, y_{m-1}, y_m) \\
&\leq 2[\lambda^k + \lambda^{k+1} + \cdots + \lambda^{m-2}]S(y_0, y_0, y_1) + \lambda^{m-1}S(y_0, y_0, y_1) \\
&= 2\lambda^k[1 + \lambda + \lambda^2 + \cdots + \lambda^{m-k-2}]S(y_0, y_0, y_1) + \lambda^{m-k-1}S(y_0, y_0, y_1) \\
&\leq 2\lambda^k[1 + \lambda + \lambda^2 + \lambda^3 + \cdots]S(y_0, y_0, y_1) \\
&\leq 2\frac{\lambda^k}{1-\lambda}S(y_0, y_0, y_1) \qquad (3.4)
\end{aligned}$$

where $\lambda = \frac{b+c}{1-a-b}$. As $k, m \to \infty$ in inequality (3.6), we obtain $S(y_k, y_k, y_m) = 0$. This shows that $\{y_k\} \subset X$ is a Cauchy sequence and then $y_k \to u \in X$ by its completeness. Besides, the continuity of F implies that

$$Fu = F(y_k) = Fy_k = y_{k+1} = u$$

Therefore, u is a fixed point of F in X.

Extracting the continuity of a map F in Theorem 3.1, we have the below result.

Theorem 3.2 If X has an ordered complete (OC) property in Theorem 3.1, then a non-decreasing mapping F has a fixed point in X.

Proof We only claim that $Fu = u$. By an ordered complete metrical property of X, we have $u = sup\, sup\, \{y_k\}$, for $k \in N$ as $y_k \to u \in X$ is a non-decreasing sequence. The non-decreasing property of a map F implies that $Fy_k \preccurlyeq Fu$ or, equivalently, $y_{k+1} \preccurlyeq Fu$, for $k \geq 0$. Since, $y_0 \prec y_1 \preccurlyeq Fu$ and $u = sup\, sup\, \{y_k\}$ as a result, we get $u \preccurlyeq Fu$. Assume $u \prec Fu$. From Theorem 3.1, there is a non-decreasing sequence $F^k u \in X$ with $F^k u = \varepsilon \in X$. Again, by an ordered complete (OC) property of X, we obtain that $\varepsilon = sup\, sup\, \{F^k u\}$. Furthermore, $y_k = F^k y_0 \preccurlyeq F^k u$, for $k \geq 1$ as a result, $y_k \prec F^k u$, for $k \geq 1$, since $y_k \preccurlyeq u \prec Fu \preccurlyeq F^k u$, for $k \geq 1$ whereas y_k and $F^k u$, for $k \geq 1$ are distinct and comparable.

Now we have the discussion below in the subsequent cases.

Case-1 If $S(y_k, y_k, F^k u) \neq 0$, then (3.1) becomes,

$$\begin{aligned}
S(y_{k+1}, y_{k+1}, F^{k+1}u) &= S\left(Fy_k, Fy_k, F(F^k u)\right) \\
&\leq a\frac{S(y_k, y_k, Fy_k)S(F^k u, F^k u, F^{k+1}u)}{S(y_k, y_k, F^k u)} \\
&\quad + b[S(y_k, y_k, Fy_k) + S(F^k u, F^k u, F^{k+1}u)] \\
&\quad + cS(y_k, y_k, F^k u) \\
&= a\frac{S(y_k, y_k, y_{k+1})S(F^k u, F^k u, F^{k+1}u)}{S(y_k, y_k, F^k u)} \\
&\quad + b[S(y_k, y_k, y_{k+1}) + S(F^k u, F^k u, F^{k+1}u)] + cS(y_k, y_k, F^k u) \qquad (3.5)
\end{aligned}$$

As $k \to \infty$ in (3.5), we get

$$S(u,u,\varepsilon) \leq cS(u,u,\varepsilon)$$

as a result, we have, $S(u,u,\varepsilon)=0$. Hence $u=\varepsilon$. In particular, $u=\varepsilon= sup\ sup\ \{F^k u\}$ in consequence, we get $Fu \preccurlyeq u$, a contradiction. Therefore,$Fu = u$.

Case-2 Case-1 If $S(y_k, y_k, F^k u) = 0$, then, $S(u,u,\varepsilon)=0$ as $k \to \infty$. By following the similar argument in Case 1, we get $Fu = u$.

Corollary 3.1 Let $(X,S,\preccurlyeq)$ be a complete partially ordred S-metric space. Suppose a self map F on X is continuous, non-decreasing and satisfies the contraction condition

$$S(Fy,Fy,Fz) \leq a\frac{S(y,y,Fy)S(z,z,Fz)}{S(y,y,z)} + bS(y,y,Fy) + cS(y,y,z) \tag{3.6}$$

for any $y \neq z \in X$ with $y \preccurlyeq z$, where $a,b,c \in [0,1)$ with $0 \leq a+b+c < 1$. If $y_0 \preccurlyeq Fy_0$ for certain $y_0 \in X$, then F has a fixed point.

Proof. It follows by $L = 0$ in Theorem 3.1.

Corollary 3.2 Let $(X,S,\preccurlyeq)$ be a complete partially ordred S-metric space. Suppose a self map F on X is continuous, non-decreasing and satisfies the contraction condition

$$S(Fy,Fy,Fz) \leq a\frac{S(y,y,Fy)S(z,z,Fz)}{S(y,y,z)} + cS(y,y,z) \tag{3.7}$$

for any $y \neq z \in X$ with $y \preccurlyeq z$, where $L \geq 0$, and $a,b,c \in [0,1)$ with $0 \leq a + 2b + c < 1$. If $y_0 \preccurlyeq Fy_0$ for certain $y_0 \in X$, then F has a fixed point.

Proof. Taking $b = 0, L = 0$ in Theorem 3.1, we obtain the desired result.

We conclude with an example.

Example 3.1 Let $(R,S,\preccurlyeq)$ be a totally ordered complete S-metric space with S-metric defined as in Example 2.8 (a). Let $F: R \to R$ be a map defined by $F(y) = \frac{3y+24n-3}{24n}$ for all $n \geq 1$. It is evident that F is continuous and non-decreasing in R and $y_0 = 0 \in R$ such that $y_0 = 0 \preccurlyeq Fy_0$. Taking $a = 0, b = 0, c = \frac{1}{n}$. For $y \preccurlyeq z$, we have

$$\begin{aligned}
S(Fy,Fy,Fz) &= 2|Fy - Fz| \\
&= 2\left|\frac{3y+24n-3}{24n} - \frac{3z+24n-3}{24n}\right| \\
&= 2\left|\frac{3(y-z)}{24n}\right| = \frac{1}{n}\left|\frac{y-z}{4}\right| \\
&\leq \frac{1}{n}|y-z| = \frac{1}{n}S(y,y,z) \\
&\leq a\frac{S(y,y,Fy)S(z,z,Fz)}{S(y,y,z)} + b[S(y,y,Fy) + S(z,z,Fz)] \\
&+cS(y,y,z)
\end{aligned}$$

holds for every $y,z \in R$. For $L \geq 0$ and $a,b,c \in [0,1)$ such that $0 \leq a + 2b + c < 1$, in particular, if we take $a = 0, b = 0, c = \frac{1}{n}$, then $0 \leq a + 2b + c < 1$ and $1 \in R$ is a fixed point of F as all the conditions of Theorem 3.1 are satisfied.

Conflict of Interest

No conflict of ınterest was declared by the authors.

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References:

[1] Abbas, M., Ali, B., Suleiman, Y. I., Generalized coupled common fixed-point results in partially ordered A-metric spaces, Fixed Point Theory and Applications, 64(2015). 1

[2] Banach, S, Sur les operations dans les ensembles abstrait et leur application aux equations, integrals. Fundam. Math. 3(1922), 133-181**.**

[3] Brouwer, L. E. J. (1912). Über Abbildung von Mannigfaltigkeiten. Mathematische Annalen, vol. 71, 1912, pp. 97–115

[4] Dass, BK, Gupta, S: An extension of Banach contraction principle through rational expressions. Indian J. Pure Appl. Math. 6 (1975), 1455–1458.

[5] Dung, N.V., Hieu, N.Y., Radojevic, S.: Fixed point theorems for g-monotone maps on partially ordered S-metric spaces, Filomat 28(9), (2014), 1885-1898, DOI 10.2298/FIL1409885D. 11

[6] Huang, H., Radenović, S., "Some fixed-point results of generalized lipschitz mappings on cone b-metric spaces over Banach algebras," Journal of Computational Analysis & Applications, vol. 20, pp. 566–583, 2016.

[7] Huang, L.G., Zhang, X.: Cone metric spaces and fixed-point theorems of contractive mappings. J. Math. Anal. Appl.332(2007), 1468-1476.

[8] Jaggi, DS, "Some unique fixed-point theorems," Indian Journal of Pure and Applied Mathematics, vol. 8, no. 2, pp. 223–230, 1977.

[9] M. Vijayakumar and R. Bhaskar, Controlled S-Metric Spaces an, J. Fixed Point Theory. 17 (2015), no. 3, 1–19.

[10] R. Kannan, Some Results on Fixed Points, Bull. Calcutta Math. Soc. 60 (1968), 71–76.

[11] S. Czerwik, Contraction Mapping in b-Metric Spaces, Acta Math. Inform. Univ. Ostraviensis 1 (1993), 5–11.

[12] S. K. Chatterjee, Fixed-Point Theorems, C. R. Acad. Bulgare Sci. 25 (1972), 727– 730.

[13] S. Reich, Fixed Point of Contractive Functions, Boll. Un. Mat. Ital. 4 (1972), 26– 42.

[14] Sedghi, S., Rao, K.P.R., Shobe, N.: Common fixed-point theorems for six weakly compatible mappings in D^*-metric spaces, Internat. J. Math. Math. Sci. 6(2007), 225- 237.

[15] Sedghi, S., Shobe, N., Aliouche, A.: A generalization of fixed-point theorem in S- metric spaces, Mat. Vesnik 64(2012), 258 -266.

[16] Sedghi, S., Shobe, N., Zhou, H.: A common fixed-point theorem in D^*-metric spaces, Fixed Point Theory Appl. 2007(2007), Article ID 27906, 13 pages.

[17] Seshagiri Rao N, Kalyani K. Unique fixed-point theorems in partially ordered metric spaces. Heliyon. 2020;6(11): e05563. https://doi.org/10.1016/j.heliyon.2020.e05563.

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COMMON FIXED-POINT THEOREMS FOR RATIONAL TYPE CONTRACTION IN SUPER METRIC SPACES

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Abstract

Our aim is to introduce the desire of a Parametric Super Metric and study some basic properties of Parametric Super Metric Spaces which is a generalization of Parametric and Super Matric Space We give some fixed-point results on a complete Parametric Super Metric Space. Some illustrative examples are given to show that our result are valid.

1. Introduction:

A fixed point of a function is a point that stays fixed by the application of the function. Fixed points, which can be considered as equilibrium states or solutions to equations, have important applications in many mathematical areas, such as numerical analysis, optimization, and dynamical systems. An example involves numerical methods, where finding a fixed point by iteration can also be relevant to solving the equation that one is iterating.

One of the fundamental results in the theory of metric spaces is the Banach Contraction Principle, the Contraction Mapping Theorem. Informally, it states under which conditions a mapping from a metric space into itself admits a unique fixed point. This theorem applies to many different areas: functional analysis, numerical methods, optimization-the name really is apt, being used to prove convergence of an iterative process and to guarantee existence and uniqueness of solutions of equations. Since then, the Banach contraction principle has been extraordinarily generalized (refer to [1-15]).

Then, Kannan [10] in 1968 introduced an important alteration to the theorem removing the continuity, an important step in the development of metric fixed-point theory, after which the Kannan theorem has been considered in several generalizations.

It was another prominent one put forward by Dass and Gupta [2], wherein the notion of Rational Contraction has been introduced. In contrast to the classical Banach contraction, which strictly relied on a constant contraction factor, their method allowed the contraction condition to be expressed in terms of rational functions to provide more flexibility. This very broad paradigmatic framework ensures the existence and uniqueness of fixed points using rational functions' properties and the completeness of the underlying metric space.

Also, Fulga and Karapinar [4] presented the notion of a super-metric space, which gave birth to new fixed-point theorems. Through this novel idea, some limitations such as congestion and stringent constraints encountered in previous works could be resolved.

In super metric space, we establish some common fixed-point theorems for rational contractions. These theorems expand and generalize several intriguing findings from metric fixed-point theory to the super metric setting. Furthermore, we present an example to illustrate our theorems.

2. Preliminaries:

First, we recall the basic results and definitions.

***Definition* 2.1** (see [5]) Consider X to be a non-empty set. A function $d: X \times X \to [0, +\infty)$ is considered a super metric if it fulfills the subsequent axioms:

(s1). $\forall\, x, y \in X$, if $d(x, y) = 0 \implies x = y$.

(s2). $\forall\, x, y \in X, d(x, y) = d(y, x)$.

(s3). There exists $s \geq 1$ such that for every $y \in X$, there exist distinct sequences $\{x_i\}, \{y_i\} \subset X$, with $d(x_i, y_i) \to 0$ when $i \to \infty$, such that

$$d(y_i, y) \leq s d(x_i, y)$$

The tripled (X, d, s) is called a super metric space.

***Definition* 2.2** (see [5]) A sequence $\{x_i\}$ on a super metric space (X, d, s):

1) converges to $x \in X \Leftrightarrow d(x_i, x) = 0$.
2) is a Cauchy sequence in $X \Leftrightarrow \{d(x_i, x_j): j > i\} = 0$.

***Proposition* 2.3** (see [5]) The limit of a convergent sequence is unique on a super metric space.

***Definition* 2.4** (see [5]) A super-metric space (X, d, s) is called complete iff each Cauchy sequence is convergent in X.

***Theorem* 2.5** (see [5]) Let (X, d, s) be a complete super-metric space and let $G: X \to X$ be a mapping. Suppose that $0 < c < 1$ such that

$$d(Gx, Gy) \leq c\, d(x, y)$$

for all $(x, y) \in X$. Then, G has a unique fixed point in X.

***Theorem* 2.6** (see [5]) Let (X, d, s) be a complete super metric space and $G: X \to X$ be a mapping, such that there exist $c \in [0, 1)$ and that

$$d(Gx, Gy) \leq c \left\{ d(x, y), \frac{d(x, Gx)\eta(y, Gy)}{\eta(x, y) + 1} \right\}$$

Then, G has a unique fixed point.

3. Main Results

Our first main result as follows.

Theorem 3.1 Let (X, d, s) be a complete super-metric space and let F, G be self-mappings of X. If there exist real numbers $k_1, k_2 \geq 0$ with $k_1 + 2\, k_2 < 1$ such that

$$d(Fx, Gy) \leq k_1 \frac{d(x, Fx) d(y, Gy)}{1 + d(x, y)} + k_2 [d(x, y) + d(x, Fx)] \qquad (3.1)$$

for all $x, y \in X$. Then, F and G have a unique common fixed point in X.

Proof. Let $x_0 \in X$ and we define the class of iterative sequences $\{x_i\}$ such that $x_{i+1} = Fx_i$, $x_{i+2} = Gx_{i+1}$ for all $i \in N$. Without loss of generality, we assume that $x_{i+2} \neq Gx_{i+1}$ for each nonnegative integer i. Indeed, if there exist a nonnegative integer i_0 such that $x_{i_0+2} = Gx_{i_0+1}$, then our proof of the Theorem proceeds as follows. Thus, from (3.1), we have

$$\begin{aligned}0 < d(x_{i+1}, x_{i+2}) &= d(Fx_i, Gx_{i+1})\\ &\leq k_1 \frac{d(x_i,Fx_i)d(x_{i+1},Gx_{i+1})}{1+d(x_i,x_{i+1})} + k_2[d(x_i, x_{i+1}) + d(x_i, Fx_i)]\\ &= k_1 \frac{d(x_i,x_{i+1})d(x_{i+1},x_{i+2})}{1+d(x_i,x_{i+1})} + 2k_2 d(x_i, x_{i+1})\\ &\leq k_1 d(x_{i+1}, x_{i+2}) + 2k_2 d(x_i, x_{i+1})\end{aligned}$$

The last inequality gives

$$0 < d(x_{i+1}, x_{i+2}) \leq \frac{2k_2}{1-k_1} d(x_i, x_{i+1}) = c_1\, d(x_i, x_{i+1})$$

where $c_1 = \frac{2\,k_2}{1-\,k_1}$. From this, we can write

$$0 < d(x_{i+1}, x_{i+2}) \leq c_1\, d(x_i, x_{i+1}) \leq {c_1}^2\, d(x_{i-1}, x_i) \leq \cdots \leq {c_1}^{i+1}\, d(x_0, x_1) \qquad (3.2)$$

On the other hand, one writes

$$\begin{aligned}0 < d(x_{i+1}, x_i) &= d(Fx_i, Gx_{i-1})\\ &\leq k_1 \frac{d(x_i,Fx_i)d(x_{i-1},Gx_{i-1})}{1+d(x_i,x_{i-1})} + k_2[d(x_i, x_{i-1}) + d(x_i, Fx_i)]\\ &= k_1 \frac{d(x_i,x_{i+1})d(x_{i-1},x_i)}{1+d(x_i,x_{i-1})} + k_2[d(x_i, x_{i-1}) + d(x_i, x_{i+1})]\\ &\leq (k_1 + k_2)d(x_i, x_{i+1}) + k_2 d(x_i, x_{i-1})\end{aligned}$$

which yields that,

$$0 < d(x_{i+1}, x_i) \leq \frac{k_2}{1-(k_1+k_2)} d(x_i, x_{i-1}) = c_2\, d(x_i, x_{i-1})$$

And then, we can write

$$0 < d(x_i, x_{i+1}) \leq c_2\, d(x_i, x_{i-1}) \leq {c_2}^2\, d(x_{i-1}, x_{i-2}) \leq \cdots \leq {c_2}^i\, d(x_0, x_1) \qquad (3.3)$$

Set $c = \{c_1, c_2\}$. By appealing to (3.2) and (3.3), we find that

$$0 < d(x_i, x_{i+1}) \leq c^i\, d(x_0, x_1) \qquad (3.4)$$

Taking the limit i tends to infinity in inequality (3.4), we get

$$d(x_i, x_{i+1}) = 0. \qquad (3.5)$$

In what follows, we want to show that the sequence $\{x_i\}$ is a Cauchy sequence. Now suppose that, $i, j \in N$ with $i > j$.Then from inequality (3.5) and using (s3), we get

$$sup\ sup\ d(x_i, x_{i+2}) \leq s\ sup\ sup\ d(x_{i+1}, x_{i+2}) \leq s\ sup\ sup\ \{c^{i+1}\, d(x_0, x_1)\} \quad (3.6)$$

Hence, $sup\ sup\ d(x_i, x_{i+2}) = 0$.

Similarly, we have

$$sup\ sup\ d(x_i, x_{i+3}) \leq s\ sup\ sup\ d(x_{i+2}, x_{i+3}) \leq s\ sup\ sup\ \{c^{i+2}\, d(x_0, x_1)\} \quad (3.7)$$

Inductively, one can conclude that $sup\ sup\ \{ d(x_i, x_j) : i > j\} = 0$. Thus, $\{x_i\}$ is a Cauchy sequence in a complete super-metric space (X, d, s), the sequence $\{x_i\}$ converges to $x^* \in X$ and then $d(x_i, x^*) = 0$. Further, we show that x^* is the fixed point of F and G. If not, $x^* \neq Fx^* \neq Gx^*$, and then $d(x^*, Fx^*) > 0$ and $d(x^*, Gx^*) > 0$. Note that

$$0 < d(x_{i+2}, Fx^*) = d(Fx^*, x_{i+2}) = d(Fx^*, Gx_{i+1})$$
$$\leq k_1 \frac{d(x^*, Fx^*) d(x_{i+1}, Gx_{i+1})}{1 + d(x^*, x_{i+1})} + k_2[d(x^*, x_{i+1}) + d(x^*, Fx^*)]$$
$$= k_1 \frac{d(x^*, Fx^*) d(x_{i+1}, x_{i+2})}{1 + d(x^*, x_{i+1})} + k_2[d(x^*, x_{i+1}) + d(x^*, Fx^*)]$$

Taking $i \to \infty$, we derive $sup\ sup\ d(x_{i+2}, Fx^*) \leq 0$. Thus, we have,

$$0 < d(x^*, Fx^*) \leq sup\ sup\ d(x_{i+2}, Fx^*) \leq 0 \tag{3.8}$$

and one can conclude that $d(x^*, Fx^*) = 0$, which implies that $Fx^* = x^*$.On the other hand,

$$0 < d(x_{i+2}, Gx^*) = d(Fx_{i+1}, Gx^*)$$
$$\leq k_1 \frac{d(x_{i+1}, Fx_{i+1}) d(x^*, Gx^*)}{1 + d(x_{i+1}, x^*)} + k_2[d(x_{i+1}, x^*) + d(x_{i+1}, Fx_{i+1})]$$
$$= k_1 \frac{d(x_{i+1}, x_{i+2}) d(x^*, Gx^*)}{1 + d(x_{i+1}, x^*)} + k_2[d(x_{i+1}, x^*) + d(x_{i+1}, x_{i+2})]$$

Taking $i \to \infty$, we derive $sup\ sup\ d(x_{i+2}, Gx^*) \leq 0$. Thus, we have,

$$0 < d(x^*, Gx^*) \leq sup\ sup\ d(x_{i+2}, Gx^*) \leq 0 \tag{3.9}$$

and one can conclude that $d(x^*, Fx^*) = 0$, which implies that $Gx^* = x^*$. Hence, x^* is a common fixed point of F and G. We shall now prove the uniqueness of x^*. Suppose there exists another point $y^* \in X$ such that $Fy^* = Gy^* = y^*$. Then, by inequality (3.1), we have

$$d(Fx^*, Gy^*) \leq k_1 \frac{d(x^*, Fx^*) d(y^*, G\, y^*)}{1 + d(x^*, y^*)} + k_2[d(x^*, y^*) + d(x^*, Fx^*)\ ]$$
$$\leq k_2 d(x^*, y^*) < d(x^*, y^*) \tag{3.10}$$

which is a contradiction.

If we take $F = G$ in condition (3.1), then we obtain the following corollary.

Corollary 3.2 Let (X, d, s) be a complete super-metric space and let F be a self-mapping of X. If there exist real numbers k_1, $k_2 \geq 0$ with $k_1 + 2k_2 < 1$ such that

$$d(Fx, Fy) \leq k_1 \frac{d(x, Fx) d(y, Fy)}{1 + d(x, y)} + k_2[d(x, y) + d(x, Fx)] \tag{3.11}$$

for all $x, y \in X$. Then, F has a unique fixed point in X.

If we take $k_1 = 0$ in Theorem 3.1 and Corollary 3.2, respectively, then we obtain the following corollaries.

Corollary 3.3 Let (X, d, s) be a complete super-metric space and let F, G be self-mappings of X. If there exists real number $0 \leq k_2 < 1$ such that

$$d(Fx, Gy) \leq k_2[d(x, y) + d(x, Fx)] \tag{3.12}$$

for all $x, y \in X$. Then, F and G have a unique common fixed point in X.

Corollary 3.4 Let (X, d, s) be a complete super-metric space and let F be a self-mapping of X. If there exists real number $0 \leq k_1 < 1$ such that

$$d(Fx, Fy) \leq k_1 d(x, y) \tag{3.13}$$

for all $x, y \in X$. Then, F has a unique fixed point in X.

We give an example which satisfy the conditions of Theorem 3.1.

Example 3.5 Let $s = 1$, and the function $d: [0, 1] \times [0, 1] \to [0, +\infty)$ be defined as follows:

$d(x, y) = xy$ for all $x \neq y$, and $x, y \in (0, 1)$;
$d(x, y) = 0$ for all $x = y$, and $x, y \in [0, 1]$;
$d(0, y) = d(y, 0) = y$ for all $y \in (0,1]$;
$d(1, y) = d(y, 1) = 1 - \frac{y}{2}$ for all $y \in [0,1)$;

First, we claim that d is super-metric on $[0, 1]$. We will concentrate on (s3) because (s1) and (s2) are simple to confirm. For any $y \in (0, 1)$, we can choose the sequences $\{x_i\}, \{y_i\} \subset [0, 1]$, where

$$x_i = \frac{i^2+1}{i^2+2}, \text{ and } y_i = \frac{i+1}{i^2+2}, \text{ for any } n \in N.$$

Since $$x_i = \frac{i^2+1}{i^2+2} = \frac{1+\frac{1}{i^2}}{1+\frac{2}{i^2}} = 1$$

and $$y_i = \frac{i+1}{i^2+2} = \frac{1+\frac{1}{i}}{i\left(1+\frac{2}{i^2}\right)} = 0.$$

Then, we have $$d(x_i, y_i) = x_i y_i = \frac{i^2+1}{i^2+2}\frac{i+1}{i^2+2} = \frac{1+\frac{1}{i^2}}{1+\frac{2}{i^2}}\frac{1+\frac{1}{i}}{i\left(1+\frac{2}{i^2}\right)} = 0.$$

Thus,

$$sup\ sup\ d(x_i, y) = sup\ sup\ x_i y = sup\ sup \left\{\left(\frac{i^2+1}{i^2+2}\right)y\right\}$$
$$= y\ sup\ sup \left(\frac{i^2+1}{i^2+2}\right) = y,$$
$$sup\ sup\ d(y_i, y) = sup\ sup\ y_i y = sup\ sup \left\{\left(\frac{i+1}{i^2+2}\right)y\right\}$$
$$= y\ sup\ sup \left(\frac{i+1}{i^2+2}\right) = 0,$$

Therefore, $sup\ sup\ d(y_i, y) = 0 < y = s\ sup\ sup\ d(x_i, y)$, and (s3) holds.
If $y = 0$, using the same sequences, we get

$$sup\ sup\ d(x_i, y) = sup\ sup\ x_i = sup\ sup\ \frac{i^2+1}{i^2+2} = 1,$$
$$sup\ sup\ d(y_i, y) = sup\ sup\ y_i = sup\ sup\ \frac{i+1}{i^2+2} = 0,$$

Therefore, $sup\ sup\ d(y_i, y) = 0 < 1 = s\ sup\ sup\ d(x_i, y)$, and again (s3) holds.

If $y = 1$, using choosing $x_i = \frac{i+1}{i^2+2}$, and $y_i = \frac{i+2}{i+3}$, for any $n \in N$. Then

$$x_i = \frac{i+1}{i^2+2} = 0 \text{ and } y_i = \frac{i+2}{i+3} = 1.$$

Then, we have

$$d(x_i, y_i) = x_i y_i = \frac{i+1}{i^2+2}\frac{i+2}{i+3} = 0.$$

Thus, $$sup\ sup\ d(x_i, y) = sup\ sup \left(1 - \frac{x_i}{2}\right)$$
$$= sup\ sup \left(1 - \frac{i+1}{2(i^2+2)}\right) = sup\ sup\ \frac{2i^2 - i + 3}{2(i^2+2)} = 1,$$

$$sup\ sup\ \ d(y_i, y) \ \ =sup\ sup\ \left(1-\frac{y_i}{2}\right)\ \ =sup\ sup\ \left(1-\frac{i+2}{2(i+3)}\right)$$
$$=sup\ sup\ \frac{i+4}{2(i+3)}\ \ =\frac{1}{2},$$

Therefore, $sup\ sup\ \ d(y_i, y)\ \ =\frac{1}{2}<1=s\ sup\ sup\ \ d(x_i, y)$,

and again (s3) holds. Hence, d defines a super-metric on $[0,1]$. Define mappings $F, G: [0,1] \to [0,1]$ as

$Fx = \frac{x}{4}$, if $x \in [0,1)$ and $Fx = \frac{1}{16}$, if $x = 1$,

$Gx = \frac{x}{2}$, if $x \in [0,1)$ and $Gx = \frac{1}{8}$, if $x = 1$.

Taking $k_1 = \frac{1}{9},\ k_2 = \frac{1}{3}$

We consider the following cases:

1. If $x, y \in (0,1)$, we have
$$d(Fx, Gy) = d\left(\frac{x}{4}, \frac{y}{2}\right) = \frac{xy}{8} \le \frac{1}{9}\frac{x^2y^2}{(8+xy)} + \frac{1}{3}(xy + x^2)$$
$$\le\ k_1\frac{d(x,Fx)d(y,Gy)}{1+d(x,y)} +\ k_2[d(x,y) + d(x,Fx)]$$
2. If $x = 0, y \in (0,1)$, we have
$$d(Fx, Gy) = d(F0, Gy) = d\left(0, \frac{y}{2}\right) = \frac{y}{2} \le \frac{1}{3}y + \frac{1}{9}0 + \frac{1}{9}\frac{y^2}{2} + \frac{1}{9}0$$
$$\le\ k_1\frac{d(x,Fx)d(y,Gy)}{1+d(x,y)} +\ k_2[d(x,y) + d(x,Fx)]$$
3. If $x = 0, y = 0$, or $x = 1, y = 1$, we have
$$d(Fx, Gy) = 0 \le \frac{1}{3}d(x,y) + \frac{1}{3}d(x,Fx) + \frac{1}{9}d(y,Gy) +$$
$$\frac{1}{9}\frac{d(x,Fx)d(y,Gy)}{1+d(x,y)}\ \le\ k_1\frac{d(x,Fx)d(y,Gy)}{1+d(x,y)} +\ k_2[d(x,y) + d(x,Fx)]$$
4. If $x = 0, y = 1$, we have
$$d(Fx, Gy) = d(F0, G1) = d\left(0, \frac{1}{8}\right) = \frac{1}{8}\ \le \frac{1}{3}(1) + \frac{1}{3}(0) + \frac{1}{9}\frac{1}{8} + \frac{1}{9}\frac{(0)\left(\frac{1}{8}\right)}{1+1}$$
$$=\ k_1\frac{d(x,Fx)d(y,Gy)}{1+d(x,y)} +\ k_2[d(x,y) + d(x,Fx)]$$
5. If $x = 1, y \in (0,1)$, we have
$$d(Fx, Gy) = d(F1, Gy) = d\left(\frac{1}{16}, \frac{y}{2}\right) = \frac{y}{32} \le \frac{1}{3}y + \frac{1}{3}\frac{1}{16} + \frac{1}{9}\frac{y^2}{2} + \frac{1}{9}\frac{\frac{y^2}{32}}{1+y}$$
$$\le\ k_1\frac{d(x,Fx)d(y,Gy)}{1+d(x,y)} +\ k_2[d(x,y) + d(x,Fx)]$$

In view of Theorem 3.1, we conclude that F and G have a unique common fixed point $0 \in [0,1]$.

References:

[1] A. Amini-Harandi and M. Fakhar, Parametric metric spaces and fixed-point theorems, Fixed Point Theory Appl. (2012).

[2] B. K. Dass and S. Gupta, An extension of Banach contraction principle through rational expressions, Indian J. Pure Appl. Math. 6 (1975), 1455–1458.
[3] E. Karapinar, A. Fulga, and H. Aydi, Study on Pata E-contractions, Adv. Differ. Equ. 2020, (2020), 539, 4.
[4] E. Karapinar and A. Fulga, Contraction in rational forms in the framework of super metric spaces, MDPI Mathematics 10 (2022), 3077, 1–12.
[5] E. Karapinar and F. Khojasteh, Super metric spaces and their applications in fixed-point theory, J. Fixed Point Theory Appl. (2021).
[6] H. Huang, Y. M. Singh, M. S. Khan, and S. Radenovic, Rational type contractions in extended b-metric spaces, Symmetry 13 (2021), 614.
[7] L. E. J. Brouwer, Über Abbildung von Mannigfaltigkeiten, Math. Ann. 71 (1912), 97–115.
[8] M. Fréchet, Sur quelques points du calcul fonctionnel, Rend. Circ. Mat. Palermo 22 (1906), 1–74.
[9] Poincaré, Henri. Sur les courbes définies par les équations différentielles. J. Math. Pures Appl. 2 (1886), 151–217.
[10] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc. 60 (1968), 71–76
[11] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux equations intégrales, Fund. Math. 3 (1922), 133–181.
[12] S. Czerwik, Contraction mapping in b-metric spaces, Acta Math. Inform. Univ. Ostraviensis 1 (1993), 5–11.
[13] S. G. Matthews, Partial metric topology, Ann. N.Y. Acad. Sci. 728 (1994), 183–197.
[14] S. K. Chatterjea, Fixed-point theorems, C. R. Acad. Bulgare Sci. 25 (1972), 727–730.
[15] Schauder, J. Der Fixpunktsatz in Funktionalräumen. Studia Math. 2 (1930), no. 1, 171–180.

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A FIXED POINT RESULT IN A CLASS OF GENERALIZED METRIC SPACES

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Abstract

In this paper, we establish a fixed point theorem for contraction-type mappings within the framework of generalized metric spaces. The proposed result significantly extends and unifies several existing fixed point theorems by relaxing conventional contractive conditions and broadening the scope of applicability in nonstandard metric structures.

Keywords: Generalized metric space, T-Orbitally complete.
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1. Introduction

The notion of a generalized metric space, introduced by Branciari [4], replaces the standard triangle inequality of a metric space with a more general inequality involving four points instead of three. Every metric space is a generalized metric space, but the converse does not hold Branciari [4]. Within this framework, Branciari established the well-known Banach fixed point theorem, and subsequent studies have further generalized his result. This naturally raises the question of whether other classical fixed point theorems—particularly those not strictly dependent on contraction mappings can also be extended to generalized metric spaces. In this paper, we explore this possibility by focusing on such results and establishing a fixed point theorem that extends existing results through slight yet meaningful generalizations suited for these broader spaces.

Specifically, we generalize the result of Saluja [14], by employing a more flexible inequality condition, thereby broadening the applicability of fixed point results in the context of generalized metric spaces.

2. Preliminaries

Following definitions are required in the sequel.

Throughout, the letters R and N will denote the set of all non-negative real numbers and the set of all positive integers respectively.

Definition 2.1: Let X be a set and $\boldsymbol{d}: \boldsymbol{X^2} \rightarrow \boldsymbol{R^+}$ a mapping such that for all $\boldsymbol{x}$, yϵX and there exist a point conditions. X. different from x and y, one has following

(i) $d(x, y) = 0$ if and only if $x = y$
(ii) $d(x, y) = d(y. x)$
(iii)$d(x, y) \leq d(x. z) + d(z. y)$

Then we will say that (X, d) is a metric space.

Definition 2.2: Let X be a set and $d: X^2 \rightarrow R^+$ a mapping such that for all $x, y \epsilon X$ and for all distinct point $x_1, x_2, x_3, \ldots\ldots, x_n \epsilon X$. cach of them different from x and y. one has

(i) $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y.x)$

(iii) $d(x, y) \leq d(x.x_1) + d(x_1, x_2) + \ldots + d(x_{n-1}, x_n) + d(x_n, y)$ Then we will say that $(X.d)$ is generalized metric space (or shortly g.m.s.)

Definition 2.3: 1.et $(X.d)$ be a g.m.s. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for any $\omega > 0$ there exist $n \in N$ such that for all $m, n\epsilon N$, $n \leq m$, one has $d(x_n, x_{n+m}) < \varepsilon$. Then $(X.d)$ is called complete if every Cauchy sequence in X is convergent in X.

Let T:X→ X be a mapping where X is a g. m. s. for each x ∈ X

$O(x, \infty) = \{x, Tx, T^2x \ldots.\}$

Definition 2.4: X is said to be T-orbitally complete if and only if every Cauchy sequence which is contained in $O(x, \infty)$ for some $x\epsilon X$ converges in X.

Definition 2.4: X is said to be T-orbitally complete if and only if every Cauchy sequence which is contained in $O(x, \infty)$ for some $x\epsilon X$ converges in X.

3. Main Result

Theorem 3.1 *Let* (X, d) *be a metric space. If* $T: X \rightarrow X$ *is a mapping such that*

$$d(Tx, Ty) \leq \beta \left[d(x, Tx) + d(y, Ty) + d(x, Ty) + d(y, Tx) + \frac{d(x,y)[1+d(x,Ty)]}{1+d(x,Ty)} \right] \quad \ldots(3.1.1)$$

$$holds\ for\ all\ x, y \in X\ where\ 0 < \beta < \frac{1}{5},$$

and if X is T − orbitally complete, then T has a unique fixed point in X.

Proof:

Let $x \in X$. *Now using* (3.1.1) *with* $y = Tx$: $d(Tx, T^2x)$

$$\leq \beta \left[d(x, Tx) + d(Tx, T^2x) + d(x, T^2x) + \frac{d(x, Tx)[1 + d(x, T^2x)]}{1 + d(x, T^2x)} \right] \Rightarrow d(Tx, T^2x)(1 - \beta)$$

$$\leq \beta[2d(x, Tx) + d(x, T^2x)] \Rightarrow d(Tx, T^2x)(1 - 2\beta)$$

$$\leq 3\beta\, d(x, Tx)$$

$$\Rightarrow d(Tx, T^2x)) \leq \left(\frac{3\beta}{(1-2\beta)}\right) d(x, Tx) \quad \ldots (3.1.2)$$

Again using (3.1.1) *for* T^2x *and* T^3x:

$$d(T^2x, T^3x) \leq \beta \left[d(Tx, T^2x) + d(T^2x, T^3x) + d(Tx, T^3x) + \frac{d(Tx, T^2x)[1 + d(Tx, T^3x)]}{1 + d(Tx, T^3x)} \right]$$

$\Rightarrow\ d(T^2x, T^3x)(1-\beta) \le \beta[2d(Tx,T^2x) + d(Tx,T^3x)]$
$\Rightarrow\ d(T^2x, T^3x)(1-2\beta) \le 3\beta\, d(Tx,T^2x) \Rightarrow d(T^2x,T^3x)$
$\le \left(\frac{3\beta}{(1-2\beta)}\right) d(Tx,T^2x)$ *Using* (3.1.2) *again*: $d(T^2x,T^3x)$
$\le \left(\frac{3\beta}{1-2\beta}\right)^2 d(x,Tx)$ .... (3.3.1) *By induction, we get*: $d(T^nx, T^{n+1}x)$
$\le r^n\, d(x,Tx)$ *where* $r = \left(\frac{3\beta}{1-2\beta}\right)$
< 1 *Hence*, $\{T^nx\}$ *is a Cauchy sequence. Let* $m > n$, *then*: $d(T^nx, T^mx)$
$\le d(T^nx,T^{n+1}x) + d(T^{n+1}x,T^{n+2}x) + \dots + d(T^{m-1}x,T^mx)$
$\le d(x,Tx)(r^n + r^{n+1} + \dots + r^{m-1})$ *Let* $m = n + p$, *then*: $d(T^nx,T^mx)$
$\le d(x,Tx)[r^n + r^{n+1} + \dots + r^{n+p-1}]$
$\le d(x,Tx) * r^n * \left[\frac{(1-r^p)}{(1-r)}\right] \quad \le d(x,Tx) * \left(\frac{r^n}{1-r}\right)$ *As* $r^n \to 0, n$
$\to \infty$, $\{T^nx\}$ *is a Cauchy sequence. Since* X *is* $T-$ *orbitally complete, let* T^nx
$= u$. *Now using* (3.1.1): $d(u,Tu)$
$\le d(u,T^nx) + d(T^nx,T^{n+1}x) + \dots + d(T^{2n-1}x,T^{2n}x) + d(T^{2n}x,Tu)$
Again by (3.1.1): $d(u,Tu)$
$\le d(u,T^nx) + \dots + \beta[d(T^{2n-1}x,T^{2n}x) + d(u,Tu) + d(T^{n-1}x,u)]$
$\Rightarrow d(u,Tu)(1-\beta) \le d(u,T^nx) + \dots + \beta\, d(T^{n-1}x,u)$ *Taking limit* n
$\to \infty$: $d(u,Tu) \le 0 \Rightarrow u$
$= Tu$ *Hence*, u *is a fixed point of* T. *Uniqueness*: *Let* v *be another fixed point, i.e.,* Tv
$= v$. *Using* (1) *with* $x = u$ *and* $y = v$

$$d(Tu,Ty) \le \beta\left[d(u,Tu) + d(v,Tv) + d(u,Tv) + d(v,Tu) + \frac{d(u,v)[1+d(u,Tv)]}{1+d(u,Tv)}\right]$$

$$d(u,v) = d(Tu,Tv) \le \beta[d(u,u) + d(v,v) + d(u,v) + d(u,v) + d(u,v)]$$

$$= 3\beta\, d(u,v) \Rightarrow d(u,v)(1-3\beta) \le 0 \text{ Since } 0 < \beta < \frac{1}{5}$$

$$\Rightarrow d(u,v) = 0 \Rightarrow u = v$$ *Hence, the fixed point is unique.*

Corollary 3.2. *Let* (X,d) *be a metric space, If* $T: X \to X$ *be a mapping such that*

$$d(Tx,Ty) \le \beta[d(x,Tx) + d(y,Ty) + d(x,Ty)]$$

holds for all $x, y \epsilon X$ *where* $0 < \beta < \frac{1}{5}$

and if X *is* $T-$ *orbitally complete then* T *has a unique fixed point in* X.

References:

[1] Abtahi, M. (2015). Fixed point theorems of Ćirić-Matkowski type in generalized metric spaces.
[2] Ahmed, M., & Radenovic, S. (2024). New fixed point results in generalized metric spaces via altering distance functions. Journal of Fixed Point Theory and Applications, 26(3).

[3] Azam, A., & Arshad, M. (2008). Kannan fixed point theorem on generalized metric spaces. Journal of Nonlinear Sciences and Applications, 1(1),pp 45–48.
[4] Branciari, A. (2000). A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces. Publ. Math. Debrecen, 57(1–2), pp 31–37.
[5] Ćirić, L. J. B. (1971). On contraction type mappings. Matematika Balkanica, 1, pp 52–57.
[6] Ćirić, L. J. B. (1974). A generalization of Banach's contraction principle. Proceedings of the American Mathematical Society, 45(2), pp 267–273.
[7] Jleli, M., & Samet, B. (2015). A generalized metric space and related fixed point theorems. Fixed Point Theory and Algorithms for Science and Engineering, 2015(1), 2.
[8] Kannan, R. (1969). Some results on fixed points. American Mathematical Monthly, 76(4), pp 405–408.
[9] Kamran, T., Postolache, M., Uddin, F., & Ali, M. U. (2016). Fixed point theorems on generalized metric space endowed with graph. Journal of Nonlinear Sciences and Applications, **9**(6), pp 4277–4285
[10]Kikina, L., & Kikina, K. (2013). A fixed point theorem in generalized metric spaces. Demonstratio Mathematica, **46**(1),pp 1–8
[11]Miheţ, D. (2015). An altering distance function in fuzzy metric fixed point theorems. Fixed Point Theory and Algorithms for Sciences and Engineering, 2015, Article 69.
[12]Mustapha, S. (2023). A fixed point result in generalized metric space with a graph. Advances in Fixed Point Theory, 13.
[13]Pacurar, C. M., & Popescu, O. (2024). Fixed point theorem for generalized Chatterjea type mappings.
[14] Petrusel, A., & Rus, I. A. (2011). Fixed point theorems in ordered metric spaces concerning generalized distance. Fixed Point Theory and Algorithms for Sciences and Engineering, 2011, Article 30.
[15] Saluja, A. S. (2010). A fixed point theorem on a class of generalized metric spaces. Indian Journal of Mathematics and Mathematical Sciences, 6, pp 193–197.
[16] Simeon, R. (2022). A fixed point result in generalized metric spaces. The Journal of Analysis, 30, pp 1467–1473.
[17]Tiwari, B. K., & Patel, N. (2024). Common fixed point theorems in generalized metric spaces using weak contractions. Communications in Mathematics and Applications, 16(1), pp59–70.

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FIXED POINT RESULTS FOR RATIONAL TYPE CONTRACTION IN S-METRIC SPACES

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Abstract

The goal of this paper is to define rational contraction in the context of S-metric spaces and develop various fixed-point theorems in order to elaborate, generalize, and synthesize a number of previously published results. Finally, to illustrate the new theorem, an example is given.

1. Introduction:

Fixed point theory is crucial in science and mathematics. This topic has drawn a lot of interest from academics in the last two decades due to its wide range of applications in disciplines such as nonlinear analysis, topology, and engineering difficulties. The Banach contraction principle [6] is the starting point for most generalizations of metric fixed point theorems. It's difficult to enumerate all of this principle's generalizations. The Banach fixed-point theorem [6] ensures the existence and uniqueness of fixed points of particular self-maps of metric spaces, as well as a constructive approach for discovering them. The S-metric space was introduced by Sedghi et al. [30]. It's a three-dimensional space called the S-metric space. The concept of A-metric space was established by Abbas et al. [2], which is a generalization of S-metric space. Jaggi [2], Das and Gupta [13] discovered the fixed-point theorem for rational contractive type conditions in metric space. The goal of this section is to define rational contraction in the setting of S-metric spaces, as well as to create various fixed-point theorems to elaborate, generalize, and synthesize several previously published results. Finally, an example is given to demonstrate the new theorem.

2. Preliminaries:

Some valuable information and ideas will be presented in this section. Metric spaces are very important in mathematics and applied sciences. So, some authors have tried to give generalizations of metric spaces in several ways. Sedghi et al. [29, 31] introduced the notion of a D^*-metric space as follows.

Definition 2.1 (see [29, 31]) Let X be a non-empty set. A D^*-metric on X is a function $D^*: X^3 \to [0, +\infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$;

(D*1). $D^*(x, y, z) \geq 0$,
(D*2). $D^*(x, y, z) = 0$if and only if $x = y = z$.
(D*3). $D^*(x, y, z) = D^*\{x, y, z\}$ (Symmetry in all three variables),
(D*4). $D^*(x, y, z) \leq D^*(x, y, z) + D^*(a, z, z)$.

Then D^* is called a D^*-metric on X and (X, D^*) is called a D^*-metric space.

Definition 2.2 (see [3]) Let X be a nonempty set. A mapping $S: X^3 \rightarrow [0, +\infty)$ is called an S-metric on X if and only if for all $x, y, z, a \in X$, the following conditions hold:

(S1). $S(x, y, z) \geq 0$,
(S2). $S(x, y, z) = 0$ if and only if $x = y = z$,
(S3). $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$

The pair (X, S) is called an S-metric space.

The following is the intuitive geometric example for S-metric spaces.

Example 2.3 (see [3]) Let $X = R^2$ and d be the ordinary metric on X. Put

$$S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$$

for all $x, y, z \in X$, that is, S is the perimeter of the triangle given by x, y, z. Then S is an S-metric on X.

Example 2.4 Let $X = [1, +\infty)$. Define $S: X^3 \rightarrow [0, +\infty)$ by

$$S(y_1, y_2, y_3) = \sum_{i=1}^{3} \quad \sum_{i<j} \quad \left|y_i - y_j\right|$$

for all $y_i \in X, i = 1,2,3$.

Lemma 2.5 (see [3]) Let (X, S) be an S-metric space. Then for all $x, y \in X$,

$$S(x, x, y) = S(y, y, x).$$

Lemma 2.6 Let (X, S) be an S-metric space. Then for all $x, y, z \in X$,

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z) \text{ and}$$
$$S(x, x, z) \leq 2S(x, x, y) + S(z, z, y).$$

Definition 2.7 (see [3]) Let X be an S-metric space.

(i). A sequence $\{y_n\}$ converges to y if and only if $S(y_n, y_n, y) = 0$. That is for each $\epsilon > 0$ there exists $n_0 \in N$ such that for all $n \geq n_0, S(y_n, y_n, y) < \epsilon$ and we denote this by

$$y_n = y.$$

(ii). A sequence $\{y_n\}$ is called a Cauchy if $S(y_n, y_n, y_m) = 0$. That is, for each $\epsilon > 0$ there exists $n_0 \in N$ such that for all $n, m \geq n_0, S(y_n, y_n, y_m) < \epsilon$.

(iii). X is called complete if every Cauchy sequence in X is a convergent.

From (see [3]), we have the following.

Example 2.8

(a). Let R be the real line. Then

$$S(x, y, z) = |x - z| + |y - z|$$

for all $x, y, z \in R$, is an S-metric on R. This S-metric is called the usual S-metric on R. Furthermore, the usual S-metric space R is complete.

(b). Let X be a non-empty set of R. Then

$$S(x, y, z) = |x - z| + |y - z|$$

for all $x, y, z \in X$, is an S-metric on X. If X is a closed subset of the usual metric space R, then the S-metric space X is complete.

Lemma 2.9 (see [3]) Let (X, S) be an S-metric space. If the sequence $\{y_n\}$ in X converges to y, then y is unique.

Lemma 2.10 (see [3]) Let (X, S) be an S-metric space. If

$$y_n = y \text{ and } z_n = z.$$

Then $$S(y_n, y_n, z_n) = S(y, y, z).$$

Remark 2.11 It is easy to see that every D^*-metric is S-metric, but in general the converse is not true, see the following example.

Example 2.12 Let $X = R^n$ and $\|\,.\,\|$ a norm on X, then

$$S(x, y, z) = \|y + z - 2x\| + \|y - z\|$$

is S-metric on X, but it is not D^*-metric because it is not symmetric.

The following lemma shows that every metric space is an S-metric space.

Lemma 2.13 Let (X, d) be a metric space. Then we have

(1). $S_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$ is an S-metric on X.
(2). $x_n = x$ in (X, d) if and only if $x_n = x$ in (X, S_d).
(3). $\{x_n\}_{n=1}^{\infty}$ is Cauchy in (X, d) if and only if $\{x_n\}_{n=1}^{\infty}$ is Cauchy in (X, S_d).
(4). (X, d) is complete if and only if (X, S_d) is complete.

Example 2.14 Let $X = R$ and let

$$S(x, y, z) = |y + z - 2x| + |y - z|$$

for all $x, y, z \in X$. By ([3]), (X, S) is an S-metric space. Dung et al. [15] proved that there does not exist any metric d such that

$$S(x, y, z) = d(x, z) + d(y, z)$$

for all $x, y, z \in X$. Indeed, suppose to the contrary that there exists a metric d with

$$S(x, y, z) = d(x, z) + d(y, z)$$

for all $x, y, z \in X$. Then

$$d(x, z) = \frac{1}{2}S(x, x, z) = 2|x - z| \text{ and}$$

$$d(x, y) = \frac{1}{2}S(x, y, y) = 2|x - y|$$

for all $x, y, z \in X$. It is a contradiction.

In 2012, Sedghi et al. [30] asserted that an S-metric is a generalization of a G-metric, that is, every G-metric is an S-metric, see [30, Remarks 1.3] and [30, Remarks 2.2]. The Example 2.1 and Example 2.2 of Dung et al. shows that this assertion is not correct. Moreover, the class of all S-metrics and the class of all G-metrics are distinct.

Definition 2.15 (see [13]) Let $(X, \preccurlyeq)$ be a partially ordered set and let $T: X \to X$ be a mapping. Then

1. elements $y, z \in X$ are comparable, if $y \preceq z$ or $z \preceq y$ holds;
2. a non-empty set X is called well ordered set, if every two elements of it are comparable;
3. T is said to be monotone non-decreasing w.r.t. $\preceq$, if for all $y, z \in X$, $y \preceq z$ implies $Ty \preceq Tz$;
4. T is said to be monotone non-increasing w.r.t. $\preceq$, if for all $y, z \in X$, $y \preceq z$ implies $Ty \succeq Tz$.

3. Main Results

First, we introduce following definitions.

Definition 3.1 The triple $(X, S, \preccurlyeq)$ is called partially ordered S-metric spaces if $(X, \preccurlyeq)$ could be a partial ordered set and (X, S) be a S-metric space.

Definition 3.2 If X is complete S-metric, then $(X, S, \preccurlyeq)$ is called complete partially ordered metric space.

Definition 3.3 A partially ordered S-metric space $(X, S, \preccurlyeq)$ is called an ordered complete (OC), if for each convergent sequence $\{y_k\} \subset X$, the subsequent condition holds: either

- if $\{y_k\} \subset X$ is a non-increasing sequence such that $y_k \to y \in X$, then $y_k \preccurlyeq y$, for all $k \in N$, that is, $y = inf\ inf\ \{y_k\}$, or
- if $\{y_k\} \subset X$ is a non-decreasing sequence such that $y_k \to y$ implies that $y_k \preccurlyeq y$, for all $k \in N$, that is, $y = sup\ sup\ \{y_k\}$.

The following is our first main outcome.

Theorem 3.4 Let $(X, S, \preccurlyeq)$ be a complete partially ordred S-metric space. Suppose a self map T on X is continuous, non-decreasing and satisfies the contraction condition

$$S(Ty, Ty, Tz) \leq a\frac{S(y,y,Ty)S(z,z,Tz)}{S(y,y,z)} + bS(y, y, Ty) + cS(z, z, Tz) + dS(y, y, z) \quad (1.3.1)$$

for any $y \neq z \in X$ with $y \preccurlyeq z$, where $a, b, c, d \in [0,1)$ with

$$0 \leq a + b + c + d < 1.$$

If $y_0 \preccurlyeq Ty_0$ for certain $y_0 \in X$, then T has a fixed point.

Proof Let $y_0 \in X$ be arbitrary and define a sequence $\{y_k\}$ by $y_{k+1} = Ty_k$. If $y_{k_0+1} = y_{k_0}$ for certain $k_0 \in N$, then y_{k_0} is a fixed point T. Assume that $y_{k+1} \neq y_k$ for each k. But $y_0 \preccurlyeq Ty_0$ and T is non-decreasing as by induction we obtain that

$$y_0 \preccurlyeq y_1 \preccurlyeq y_2 \preccurlyeq \cdots \preccurlyeq y_k \preccurlyeq y_{k+1} \leq \cdots \quad (1.3.2)$$

By (1), we have

$$\begin{aligned}
S(y_{k+1}, y_{k+1}, y_k) = S(Ty_k, Ty_k, Ty_{k-1}) &\leq a\frac{S(y_k,y_k,Ty_k)S(y_{k-1},y_{k-1},Ty_{k-1})}{S(y_k,y_k,y_{k-1})} \\
&+ bS(y_k, y_k, Ty_k) + cS(y_{k-1}, y_{k-1}, Ty_{k-1}) + dS(y_k, y_k, y_{k-1}) \\
&= a\frac{S(y_k,y_k,y_{k+1})S(y_{k-1},y_{k-1},y_k)}{S(y_k,y_k,y_{k-1})} \\
&+ bS(y_k, y_k, y_{k+1}) + cS(y_{k-1}, y_{k-1}, y_k) + dS(y_k, y_k, y_{k-1}) \\
&= a\frac{S(y_{k+1},y_{k+1},y_k)S(y_k,y_k,y_{k-1})}{S(y_k,y_k,y_{k-1})} \\
&+ bS(y_{k+1}, y_{k+1}, y_k) + cS(y_k, y_k, y_{k-1}) + dS(y_k, y_k, y_{k-1}) \\
&= (a + b)S(y_{k+1}, y_{k+1}, y_k) + (c + d)S(y_k, y_k, y_{k-1})
\end{aligned}$$

which infer that

$$\begin{aligned}
S(y_{k+1}, y_{k+1}, y_k) &\leq \left(\frac{c+d}{1-a-b}\right) S(y_k, y_k, y_{k-1}) \\
&\leq \left(\frac{c+d}{1-a-b}\right)^k S(y_1, y_1, y_0) \leq \cdots \quad (1.3.3)
\end{aligned}$$

For $m, k \in N$ with $m > k$, by repeated use of (S3), we have

$$S(y_k, y_k, y_m) \le 2S(y_k, y_k, y_{k+1}) + S(y_m, y_m, y_{k+1})$$
$$\le 2S(y_k, y_k, y_{k+1}) + S(y_{k+1}, y_{k+1}, y_m)$$
$$\le 2S(y_k, y_k, y_{k+1}) + 2S(y_{k+1}, y_{k+1}, y_{k+2}) + S(y_m, y_m, y_{k+2})$$
$$\le 2S(y_k, y_k, y_{k+1}) + 2S(y_{k+1}, y_{k+1}, y_{k+2}) + S(y_{k+2}, y_{k+2}, Sy_m)$$
$$\le 2S(y_k, y_k, y_{k+1}) + 2S(y_{k+1}, y_{k+1}, y_{k+2})$$
$$+ 2S(y_{k+2}, y_{k+2}, y_{k+3}) + S(y_m, y_m, y_{k+3})$$
$$\le 2S(y_k, y_k, y_{k+1}) + 2S(y_{k+1}, y_{k+1}, y_{k+2}) + 2S(y_{k+2}, y_{k+2}, y_{k+3})$$
$$+ 2S(y_{k+3}, y_{k+3}, y_{k+4}) + \cdots + 2S(y_{m-2}, y_{m-2}, y_{m-1}) + S(y_{m-1}, y_{m-1}, y_m)$$
$$\le 2[\lambda^k + \lambda^{k+1} + \cdots + \lambda^{m-2}]S(y_0, y_0, y_1) + \lambda^{m-1}S(y_0, y_0, y_1)$$
$$= 2\lambda^k[1 + \lambda + \lambda^2 + \cdots + \lambda^{m-k-2}]S(y_0, y_0, y_1) + \lambda^{m-k-1}S(y_0, y_0, y_1)$$
$$\le 2\lambda^k[1 + \lambda + \lambda^2 + \lambda^3 + \cdots]S(y_0, y_0, y_1)$$
$$\le 2\frac{\lambda^k}{1-\lambda}S(y_0, y_0, y_1) \quad (1.3.4)$$

where $\lambda = \frac{c+d}{1-a-b}$. As $k, m \to \infty$ in inequality (1.3.6), we obtain $S(y_k, y_k, y_m) = 0$. This shows that $\{y_k\} \subset X$ is a Cauchy sequence and then $y_k \to y \in X$ by its completeness. Besides, the continuity of T implies that

$$Ty = T(y_k) = Ty_k = y_{k+1} = y$$

Therefore, y is a fixed point of T in X.

Extracting the continuity of a map T in Theorem 1 of 1.2.3, we have the below result.

Theorem 3.5 If X has an ordered complete (OC) property in Theorem 1 of 1.2.3, then a non-decreasing mapping T has a fixed point in X.

Proof We only claim that $Ty = y$. By an ordered complete metrical property of X, we have $y = sup\, sup\,\{y_k\}$, for $k \in N$ as $y_k \to y \in X$ is a non-decreasing sequence. The non-decreasing property of a map T implies that $Ty_k \preccurlyeq Ty$ or, equivalently, $y_{k+1} \preccurlyeq Ty$, for $k \ge 0$. Since, $y_0 \prec y_1 \preccurlyeq Ty$ and $y = sup\, sup\,\{y_k\}$ as a result, we get $y \preccurlyeq Ty$. Assume $y \prec Ty$. From Theorem 1 of 1.2.3, there is a non-decreasing sequence $T^k y \in X$ with $T^k y = \varepsilon \in X$. Again, by an ordered complete (OC) property of X, we obtain that $\varepsilon = sup\, sup\,\{T^k y\}$. Furthermore, $y_k = T^k y_0 \preccurlyeq T^k y$, for $k \ge 1$ as a result, $y_k \prec T^k y$, for $k \ge 1$, since $y_k \preccurlyeq y \prec Ty \preccurlyeq T^k y$, for $k \ge 1$ whereas y_k and $T^k y$, for $k \ge 1$ are distinct and comparable. Now we have the discussion below in the subsequent cases.

Case-1 If $S(y_k, y_k, T^k y) \neq 0$, then (1) becomes,

$$S(y_{k+1}, y_{k+1}, T^{k+1}y) = S\left(Ty_k, Ty_k, T(T^k y)\right)$$
$$\le a\frac{S(y_k, y_k, Ty_k)S(T^k y, T^k y, T^{k+1}y)}{S(y_k, y_k, T^k y)}$$
$$+ bS(y_k, y_k, Ty_k) + cS(T^k y, T^k y, T^{k+1}y) + dS(y_k, y_k, T^k y)$$
$$= a\frac{S(y_k, y_k, y_{k+1})S(T^k y, T^k y, T^{k+1}y)}{S(y_k, y_k, T^k y)}$$
$$+ bS(y_k, y_k, y_{k+1}) + cS(T^k y, T^k y, T^{k+1}y)$$
$$+ dS(y_k, y_k, T^k y) \quad (1.3.5)$$

As $k \to \infty$ in (1.3.5), we get

$$S(y, y, \varepsilon) \leq dS(y, y, \varepsilon)$$

as a result, we have, $S(y, y, \varepsilon) = 0$. Hence $y = \varepsilon$. In particular, $y = \varepsilon = sup\ sup\ \{T^k y\}$ in consequence, we get $Ty \preccurlyeq y$, a contradiction. Therefore, $Ty = y$.

Case-2 If $S(y_k, y_k, T^k y) = 0$, then, $S(y, y, \varepsilon) = 0$ as $k \to \infty$. By following the similar argument in Case 1, we get $Ty = y$.

Corollary 1 Let $(X, S, \preccurlyeq)$ be a complete partially ordred S-metric space. Suppose a self map T on X is continuous, non-decreasing and satisfies the contraction condition

$$S(Ty, Ty, Tz) \leq a\frac{S(y,y,Ty)S(z,z,Tz)}{S(y,y,z)} + b[S(y, y, Ty) + S(z, z, Tz)] + dS(y, y, z) \quad (1.3.6)$$

for any $y \neq z \in X$ with $y \preccurlyeq z$, where $a, b, c \in [0,1)$ with $0 \leq a + 2b + d < 1$. If $y_0 \preccurlyeq Ty_0$ for certain $y_0 \in X$, then T has a fixed point.

Proof. It follows by $b = c$ in Theorem 1 of 1.2.3.

Corollary 2 Let $(X, S, \preccurlyeq)$ be a complete partially ordred S-metric space. Suppose a self map T on X is continuous, non-decreasing and satisfies the contraction condition

$$S(Ty, Ty, Tz) \leq a\frac{S(y,y,Ty)S(z,z,Tz)}{S(y,y,z)} + dS(y, y, z) \quad (1.3.7)$$

for any $y \neq z \in X$ with $y \preccurlyeq z$, where $L \geq 0$, and $a, d \in [0,1)$ with $0 \leq a + d < 1$. If $y_0 \preccurlyeq Ty_0$ for certain $y_0 \in X$, then T has a fixed point.

Proof. Taking $b = c = 0$ in Theorem 1 of 1.2.3, we obtain the desired result. We conclude with an example.

Example 3.6 Let $(R, S, \preccurlyeq)$ be a totally ordered complete S-metric space with S-metric defined as in Example 8 (a) of 1.2.2. Let $T: R \to R$ be a map defined by $T(y) = \frac{3y+24n-3}{24n}$ for all $n \geq 1$. It is evident that T is continuous and non-decreasing in R and $y_0 = 0 \in R$ such that $y_0 = 0 \preccurlyeq Ty_0$. Taking $a = 0, b = c = 0, d = \frac{1}{n}$. For $y \preccurlyeq z$, we have

$$\begin{aligned} S(Ty, Ty, Tz) &= 2|Ty - Tz| \\ &= 2\left|\frac{3y+24n-3}{24n} - \frac{3z+24n-3}{24n}\right| \\ &= 2\left|\frac{3(y-z)}{24n}\right| = \frac{1}{n}\left|\frac{y-z}{4}\right| \\ &\leq \frac{1}{n}|y - z| = \frac{1}{n}S(y, y, z) \\ &\leq a\frac{S(y,y,Ty)S(z,z,Tz)}{S(y,y,z)} + bS(y, y, Ty) + cS(z, z, Tz) \\ &\quad + dS(y, y, z) \end{aligned}$$

holds for every $y, z \in R$. For $L \geq 0$ and $a, b, c, d \in [0, 1)$ such that $0 \leq a + b + c + d < 1$, in particular, if we take $a = 0, b = c = 0, d = \frac{1}{n}$, then $0 \leq a + b + c + d < 1$ and $1 \in R$ is a fixed point of T as all the conditions of Theorem 1 of 1.2.3 are satisfied.

References:

[1]. Abbas, M., Ali, B., Suleiman, Y. I., Generalized coupled common fixed-point results in partially ordered *A*-metric spaces. Fixed-point Theory and Applications. 64, (2015).

[2]. Abdeljawad, T., Karapinar E., Tas K.: Existence and Uniqueness of a common Fixed-point on Partial Metric Spaces. Applied Mathematics Letters. 24(11), (2011), 1900-1904.

[3]. Afshari, H., Aydi, H., Karapinar, E., On generalized α-ψ-Geraghty contractions on b-metric spaces. Georgian Mathematical Journal. 27 (1), (2020), 9–21.

[4]. Aghajani, A., Abbas, M., Roshan, R.J.: Common Fixed-point of Generalized Weak Contractive Mappings in Partially Ordered -metric Spaces. Filomat. 28(6), (2014), 1087–1101DOI 10.2298/FIL140 6087A.

[5]. Bakhtin, I. A.: The contraction mapping principle in almost metric spaces. Funct. Anal., Gos. Ped. Inst. Unianowsk 30, (1989), 26-37.

[6]. Banach, S.: Sur les operations dans les ensemles abstraits et leur application aux equations integrals. Fund. Math. 3, (1922), 133-181.

[7]. Bergiz, M., Karapinar, E., Roldan, A.: Discussion on generalized -contractive mappings via generalized altering distance function and related fixed-point theorems. Abstr. Appl. Anal. 22(2), (2014), 258-272.

[8]. Bianchini R.M.T.: Su un problema di S. Reich riguardante la teoria dei punti fissi, Bolletino U.M.I., 4(5) (1972), 103–106.

[9]. Caccioppoli, R., Un teorema generale sull'esistenza de elemente uniti in una transformazione funzionale, Rend. Acad. Naz. Linzei, 11 (1930), pp. 31–49.

[10]. Chatterjea, S.K.: Fixed point theorems, C. R. Acad. Bulgare Sci., 25, pp. 727–730, 1972.

[11]. Ciric L. B.: On some nonexpansive mappings and fixed-points. Indian J. Pure Appl.Math. 24(3), (1993), 145–149.

[12]. Czerwik, S.: Contraction mappings in b-metric spaces. Acta Mathematica et. Informatica Universitatis Ostraviensis. 1, (1993), 5–11.

[13]. Dass, B.K., Gupta, S.: An extension of Banach contraction principle through rational expressions. Indian J. Pure Appl. Math. 6, (1975), 1455-1458.

[14]. Dhage, B.C.: A study of some fixed-point theorem. Ph.D. thesis, Marathwada University, Aurangabad, India (1984).

[15]. Dung, N.V., Hieu, N.Y., Radojevic, S.: Fixed-point theorems for g-monotone maps on partially ordered S-metric spaces. Filomat. 28(9), (2014), 1885-1898, DOI 10.2298/FIL1409885D.

[16]. Fréchet, M.: Sur quelques points du calcul fonctionnel. Rendic.Circ, Mat. Palermo. 22 (1906), 1–74.

[17]. Gähler, S.: 2-metriche raume und ihre topologische strukture. Math. Nachr. 26, (1963), 115-148.

[18]. Hausdorff, F.: Grundzuge der Mengenlehre, Veit, Leipzig, (1914).
[19]. Huang, L.G., Zhang, X.: Cone metric spaces and fixed-point theorems of contractive mappings. J. Math. Anal. Appl. 332, (2007), 1468-1476.
[20]. Jungck, G.: Commuting mappings and fixed-points. Amer. Math. Monthly. 83, (1976), 261-263.
[21]. Kannan, R.: Some results on fixed points, Bull. Calcutta Math. Soc., 10 (1968), 71–76
[22]. Mustafa, Z, Sims, B.: Some results concerning D-metric spaces. In: Proceedings of the International Conferences on Fixed-point Theory and Applications. Valencia, Spain, (2003), 189-198.
[23]. Mustafa, Z.: A new structure for generalized metric spaces with applications to fixed point theory, Ph.D. thesis, The University of Newcastle, Callaghan, Australia, 2005.
[24]. Petrusel A., Rus I.A.: Fixed point theorems in ordered L-spaces, Proc. Amer. Math. Soc., 134) (2006), 411–418.
[25]. Poincare, H.: Sur less courbes defines par les iquations differentielles. Journal. De. Math. 2, (1865), 54-65.
[26]. Reich, S., Kannan's fixed-point theorem. Bollettino dell'Unione Matematica Italiana, 4, (1971), 1 –11.
[27]. Reurings, M.C.B., Ran, A.C.M.: A fixed-point theorem in partially ordered sets and some applications to matrix equations. Proceedings of the American Mathematical Society, 132, (2004), 1435–1443.
[28]. Rus I. A.: Generalized Contractions, Seminar on fixed point theory, Univ. Cluj- Napoca, Preprint 3 (1983), 1–130.
[29]. Sedghi, S., Rao, K.P.R., Shobe, N.: Common fixed-point theorems for six weakly compatible mappings in -metric spaces. Internat. J. Math. Math. Sci. 6(2007), 225-237.
[30]. Sedghi, S., Shobe, N., Aliouche, A.: A generalization of fixed-point theorem in S-metric spaces. Mat. Vesnik. 64, (2012), 258 -266.
[31]. Sedghi, S., Shobe, N., Zhou, H.: A common fixed-point theorem in -metric spaces. Fixed-point Theory Appl. (2007).
[32]. Zamfirescu T., Fixed point theorems in metric spaces, Arch. Math. (Basel) (1972), 292–298.

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SOME FIXED-POINT THEOREMS FOR EXPANSIVE MAPPINGS IN PARAMETRIC METRIC SPACES

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Abstract

In this manuscript, we present some unique fixed-point theorems satisfying expansive type conditions by considering surjective self-mapping in the context of parametric metric space.

1. INTRODUCTION:

The concept of metric spaces has been generalized in many directions. The notion of a b-metric space was studied by Czerwik in [2-3] and a lot of fixed-point results for single-valued and multi-valued mappings by many authors have been obtained in (ordered) b-metric spaces (see, e.g., [4]-[5]). The concept of fuzzy set was introduced by Zadeh [9] in 1965. In 1975, Kramosil and Michalek [7] introduced the notion of fuzzy metric space, which can be regarded as a generalization of the statistical (probabilistic) metric space. This work has provided an important basis for the construction of fixed-point theory in fuzzy metric spaces. In 2004, Park introduced the notion of intuitionistic fuzzy metric space [8].

2. DEFINITIONS AND PRELIMINARIES

Throughout this paper $\mathbb{R}$ and $\mathbb{R}^+$ will represents the set of real numbers and nonnegative real numbers, respectively.

In 2014, Hussain et al. [6] defined and studied the concept of parametric metric space as follows.

Definition 2.1 Let X be a nonempty set and $\mathcal{P} : X \times X \times (0, +\infty) \to [0, +\infty)$ be a function. We say $\mathcal{P}$ is a parametric metric on X if,

(1) $\mathcal{P}(x, y, t) = 0$ for all $t > 0$ if and only if $x = y$;

(2) $\mathcal{P}(x, y, t) = \mathcal{P}(y, x, t)$ for all $t > 0$;

(3) $\mathcal{P}(x, y, t) \leq \mathcal{P}(x, z, t) + \mathcal{P}(z, y, t)$ for all $x, y, z \in X$ and all $t > 0$:

and one says the pair $(X, \mathcal{P})$ is a parametric metric space.

The following definitions are required in the sequel which can be found in [1].

Definition 2.2 Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a parametric metric space $(X, \mathcal{P})$.

1. $\{x_n\}_{n=1}^{\infty}$ is said to be convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$, for all $t > 0$, if $\lim_{n\to\infty} \mathcal{P}(x_n, x, t) = 0$.
2. $\{x_n\}_{n=1}^{\infty}$ is said to be a Cauchy sequence in X if for all $t > 0$, if $\lim_{n,m\to\infty} \mathcal{P}(x_n, x_m, t) = 0$.

3. $(X, \mathcal{P})$ is said to be complete if every Cauchy sequence is a convergent sequence.

Definition 2.3 Let $(X, \mathcal{P})$ be a parametric metric space and $T: X \to X$ be a mapping. We say T is a continuous mapping at x in X, if for any sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} Tx_n = Tx$.

Example 2.4 Let X denote the set of all functions $f : (0, +\infty) \to \mathbb{R}$. Define $\mathcal{P} : X \times X \times (0, +\infty) \to [0, +\infty)$ by $\mathcal{P}(f, g, t) = |f(t) - g(t)|\ \forall\ f, g \in X$ and all $t > 0$. Then $\mathcal{P}$ is a parametric metric on X and the pair $(X, \mathcal{P})$ is a parametric metric space.

3. FIXED POINT RESULTS IN PARAMETRIC METRIC SPACES:

In this section, we prove some unique fixed-point theorems satisfying expansive condition by considering surjective self-mapping in the context of parametric metric space.

We begin with a simple but a useful lemma.

Lemma 3.1 Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a parametric metric space $(X, \mathcal{P})$ such that

(3.1.1) $$\mathcal{P}(x_n, x_{n+1}, t) \le \lambda^n \mathcal{P}(x_0, x_1, t)$$

where $\lambda \in [0, 1)$ and $n = 1, 2, \ldots$. Then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(X, \mathcal{P})$.

Proof Let $m > n \ge 1$. It follows that

(3.1.2) $$\begin{aligned}\mathcal{P}(x_n, x_m, t) &\le \mathcal{P}(x_n, x_{n+1}, t) + \mathcal{P}(x_{n+1}, x_{n+2}, t) + \cdots + \mathcal{P}(x_{m-1}, x_m, t)\\ &\le (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1})\, \mathcal{P}(x_0, x_1, t)\\ &\le \frac{\lambda^n}{1-\lambda}\, \mathcal{P}(x_0, x_1, t)\end{aligned}$$

for all $t > 0$. Since $\lambda < 1$. Assume that $\mathcal{P}(x_0, x_1, t) > 0$. By taking limit as $m, n \to +\infty$ in above inequality we get

(3.1.3) $$\lim_{n,m\to\infty} \mathcal{P}(x_n, x_m, t) = 0.$$

Therefore, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Also, if $\mathcal{P}(x_0, x_1, t) = 0$, then $\mathcal{P}(x_n, x_m, t) = 0$ for all $m > n$ and hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X.

Now, our first main results as follows.

Theorem 3.2 Let $(X, \mathcal{P})$ be a complete parametric metric space and $T: X \to X$ be a surjection. Suppose that there exist $a, b \ge 0$ with $a + 2b > 1$ such that

(3.2.1) $$\mathcal{P}(Tx, Ty, t) \ge a\, \mathcal{P}(x, y, t) + b\, [\mathcal{P}(x, Tx, t) + \mathcal{P}(y, Ty, t)]$$

$\forall\, x, y \in X$ with $x \ne y$ and all $t > 0$. Then T has a fixed point in X.

Proof Under the assumption. It is clear that T is injective. Let G be the inverse mapping of T . Choose $x_0 \in X$, set $x_1 = G(x_0)$, $x_2 = G(x_1) = G^2(x_0), \ldots\ldots\ldots, x_{n+1} = G(x_n) = G^{n+1}(x_0) \ldots\ldots..$. Without loss of generality, we assume that $x_{n-1} \ne x_n$ for all $n = 1,2, \ldots.$(otherwise, if there exists some n_0 such that $x_{n_0-1} = x_{n_0}$, then x_{n_0} is a fixed point of T). It follows that from condition (4.3.2.1)

(3.2.2) $$\begin{aligned}\mathcal{P}(x_{n-1}, x_n, t) &= \mathcal{P}(TT^{-1}x_{n-1}, TT^{-1}x_n, t)\\ &\ge a\, \mathcal{P}(T^{-1}x_{n-1}, T^{-1}x_n, t) + b\, \mathcal{P}(T^{-1}x_{n-1}, TT^{-1}x_{n-1}, t) + b\, \mathcal{P}(T^{-1}x_n, TT^{-1}x_n, t)\end{aligned}$$

$= a\,\mathcal{P}(Gx_{n-1}, Gx_n, t) + b\,\mathcal{P}(Gx_{n-1}, x_{n-1}, t) + b\,\mathcal{P}(Gx_n, x_n, t)$
$= a\,\mathcal{P}(x_n, x_{n+1}, t) + b\,\mathcal{P}(x_n, x_{n-1}, t) + b\mathcal{P}(x_{n+1}, x_n, t)$

Hence

(3.2.3) $$(1-b)\,\mathcal{P}(x_{n-1}, x_n, t) \geq (a+b)\,\mathcal{P}(x_{n+1}, x_n, t)$$

If $a = 0$, then $b > 0$. The above inequality implies that a negative number is greater then or equal to zero. This is impossible. So, $a \neq 0$ and$(1-2b) > 0$. Therefore,

(3.2.4) $$\mathcal{P}(x_{n+1}, x_n, t) \leq k\,\mathcal{P}(x_{n-1}, x_n, t)$$

where $k = \frac{1-b}{a+b} < 1$for all $n \in \mathbb{N} \cup \{0\}$ and $t > 0$. Repeating (4.3.2.4) n-times, we get

(3.2.5) $$\mathcal{P}(x_{n+1}, x_n, t) \leq k^n\,\mathcal{P}(x_0, x, t)$$

for all $t > 0$. By Lemma 4.3.1,$\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since $(X, \mathcal{P})$ is a complete parametric metric space, there exists $x^\star \in X$ such that $x_n \to x^\star$ as $n \to \infty$. Now since T is surjective map. So there exists a point y in X such that $x^\star = Ty$. Consider

(3.2.6) $$\begin{aligned}\mathcal{P}(x_n, x^\star, t) &= \mathcal{P}(Tx_{n+1}, Ty, t)\\ &\geq a\,\mathcal{P}(x_{n+1}, y, t) + b\,\mathcal{P}(x_{n+1}, Tx_{n+1}, t) + b\,\mathcal{P}(y, Ty, t)\\ &= a\,\mathcal{P}(x_{n+1}, y, t) + b\,\mathcal{P}(x_{n+1}, x_n, t) + b\mathcal{P}(y, x^\star, t)\end{aligned}$$

which implies that as $n \to +\infty$

(3.2.7) $$0 \geq (a+b)\,\mathcal{P}(y, x^\star, t)$$

Hence $y = x^\star$. This gives that $x^\star$ is a fixed point of T. This completes the proof.

Now we give an example illustrating Theorem 4.3.2.

Example 3.3 Let $X = [0, +\infty)$ be endowed with parametric metric,

$$\mathcal{P}(x, y, t) = \begin{cases} t\max\{x, y\}, & x \neq y \\ 0, & x = y \end{cases}$$

for all $x, y \in X$ and $t > 0$. Define $T: X \to X$ by $Tx = \frac{5}{2}x$. Obviously, T is continuous surjective map on X. So, for $a = 4, b = -2$ all the conditions of Theorem 4.3.2 are satisfied. Therefore $x^\star = 0$ is the unique fixed point of T.

Setting $b = 0$ and $a = k$ in Theorem 4.3.2, we can obtain the following result.

Corollary 3.4 Let $(X, \mathcal{P})$ be a complete parametric metric space and $T: X \to X$ be a surjection. Suppose that there exists a constant $k > 1$ such that

(3.4.1) $$\mathcal{P}(Tx, Ty, t) \geq k\mathcal{P}(x, y, t)$$

$\forall\, x, y \in X$ and all $t > 0$. Then T has a unique fixed point in X.

Proof From Theorem 3.2, it follows that T has a fixed point $x^\star$ in X by setting $b = 0$ and $a = k$ in condition (3.4).

Uniqueness. Suppose that $x^\star \neq y^\star$ is also another fixed point of T, then from condition (3.4.1), we obtain

(3.4.2) $$\begin{aligned}\mathcal{P}(x^\star, y^\star, t) &= \mathcal{P}(Tx^\star, Ty^\star, t)\\ &\geq k\mathcal{P}(x^\star, y^\star, t)\end{aligned}$$

which implies $\mathcal{P}(x^\star, y^\star, t) = 0$, that is $x^\star = y^\star$. This completes the proof.

Corollary 3.5 Let $(X, \mathcal{P})$ be a complete parametric metric space and $T: X \to X$ be a surjection. Suppose that there exists a positive integer n and a real number $k > 1$ such that

(3.5.1) $\mathcal{P}(T^n x, T^n y, t) \geq k\,\mathcal{P}(x, y, t)$

$\forall\, x, y \in X$ and all $t > 0$. Then T has a unique fixed point in X.

Proof From Corollary 3.4, T^n has a fixed point $x^\star$. But $T^n(Tx^\star) = T(T^n x^\star) = Tx^\star$, So $Tx^\star$ is also a fixed point of T^n. Hence $Tx^\star = x^\star, x^\star$ is a fixed point of T. Since the fixed point of T is also fixed point of T^n , the fixed point of T is unique.

References:

[1] Brouwer, L. E. J. (1912). Über Abbildung von Mannigfaltigkeiten. *Mathematische Annalen*, 71, 97–115.
[2] Czerwik, S. (1993). Contraction mappings in b-metric spaces. *Acta Mathematica et Informatica Universitatis Ostraviensis*, 1, 5–11.
[3] Czerwik, S. (1998). Nonlinear set-valued contraction mappings in b-metric spaces. *Atti del Seminario Matematico e Fisico dell'Università di Modena,* 46(2), 263–276.
[4] Daffer, P. Z., & Kaneko, H. (1992). On expansive mappings. *Mathematica Japonica*, 37, 733–735.
[5] Dass, B. K., & Gupta, S. (1975). An extension of Banach contraction principle through rational expression. *Indian Journal of Pure and Applied Mathematics*, 6(12), 1455–1458.
[6] Hussain, N., Khaleghizadeh, S., Salimi, P., Afrah, A. N., & Abdou, A. N. (2014). A new approach to fixed point results in triangular intuitionistic fuzzy metric spaces. *Abstract and Applied Analysis*, *2014*, Article 690139, 16 pages.
[7] Kramosil, I., & Michalek, J. (1975). Fuzzy metrics and statistical metric spaces. *Kybernetika*, 11(5), 336–344.
[8] Park, J. H. (2004). Intuitionistic fuzzy metric spaces. *Chaos, Solitons & Fractals*, 22(5), 1039–1046.
[9] Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, 8, 338–353.

SOME FIXED POINT AND PERIODIC POINT THEOREMS IN COMPLETE CONE METRIC SPACES

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Abstract:

Abbas M. and Rhoades B. E. [2] proved some fixed point theorems in cone metric spaces. In this paper we further generalize these results by introducing new contractive condition.

Keywords: Metric space, Fixed point, Complete cone metric space.
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1. Introduction

Abbas M. and Rhoades B. E. [2] extend the fixed point theory in the context of cone metric spaces, a concept introduced by Huang and Zhang [6]. Abbas and Jungck [1] demonstrated the existence of coincidence and common fixed points for mappings that fulfill specific contractive conditions within the setting of cone metric spaces. The authors [2] prove new fixed point theorems for certain types of contractive mappings in cone metric spaces. Their results do not require the assumption of continuity or commutativity of the mappings, which makes their theorems more broadly applicable. In this paper we generalize these results by using new contractive conditions and these results appear as special cases of our result.

2.Preliminaries

Consistent with Abbas M. and Rhoades B.E. [2], the following definitions are required in this context.

Definition 2.1 Let E be a real Banach space. A subset P of E is called a cone if and only if:

(a) P is closed, non empty and $P \neq \{0\}$;
(b) $a, b \in R,\ a, b \geq 0, x, y \in P$ imply that $ax + by \in P$;
(c) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K\|y\| \#(2.1)$$

The least positive number satisfying the above inequality is called the normal constant of P, while $x \ll y$ stands for $y - x \in \text{int } P$ (interior of P). We shall write $x < y$ to indicate that $x \ll y$ but $x \neq y$.

Definition 2.2 Let X be a nonempty set. Suppose that the mapping

$d: X \times X \to E$ satisfies:

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space. The concept of a cone metric space is more general than that of a metric space.

Definition 2.3 Let (X, d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in X$. For every $c \in E$ with $0 \ll c$, we say that $\{x_n\}$ is:

(e) a Cauchy sequence if there is an N such that, for all $n, m > N, d(x_n, x_m) \ll c$;

(f) a convergent sequence if there is an N such that, for all $n > N, d(x_n, x) \ll c$ for some x in X.

Definition 2.3 A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X i.e. for any sequence $\{x_n\}$ in X, $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$. The limit of a convergent sequence is unique provided P is a normal cone with normal constant K.(see [2] & [6])

3. Main Result

Theorem 3.1 Let (X, d) be a complete cone metric spaces, and P a normal cone with normal constant K. Suppose that the mapping f and g are two self-maps of X satisfying:

$$d(fx, gy) \leq \alpha d(x, y) + \beta[d(x, fx) + d(y, gy)] + \gamma[d(x, gy) + d(y, fx)] + \delta \frac{d(y, gy)[1 + d(x, fx)]}{1 + d(x, y)} \quad \ldots \quad (3.1.1)$$

$\forall\, x, y \in X$, where $\alpha, \beta, \gamma, \delta \geq 0$ and $\alpha + 2\beta + 2\gamma + \delta < 1$. Then f and g have a unique common fixed point in X. Moreover, any fixed point of f is fixed point of g and conversely.

Proof- Suppose x_0 is an arbitrary point of X, and define $\{x_n\}$ by

$$x_{2n+1} = fx_{2n} \quad , \quad x_{2n+2} = gx_{2n+1}, n = 0,1,2 \ldots$$

Now,

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(fx_{2n}, gx_{2n+1}) \\ &\leq \alpha d(x_{2n}, x_{2n+1}) + \beta[d(x_{2n}, fx_{2n}) + d(x_{2n+1}, gx_{2n+1})] + \gamma[d(x_{2n}, gx_{2n+1}) + d(x_{2n+1}, fx_{2n})] + \delta \frac{d(x_{2n+1}, gx_{2n+1})[1 + d(x_{2n}, fx_{2n})]}{1 + d(x_{2n}, x_{2n+1})} \\ &\leq \alpha d(x_{2n}, x_{2n+1}) + \beta[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + \gamma[d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})] + \delta \frac{d(x_{2n+1}, x_{2n+2})[1 + d(x_{2n}, x_{2n+1})]}{1 + d(x_{2n}, x_{2n+1})} \\ &\leq \alpha d(x_{2n}, x_{2n+1}) + \beta[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + \gamma[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + \delta d(x_{2n+1}, x_{2n+2}) \end{aligned}$$

$$d(x_{2n+1}, x_{2n+2}) \leq (\alpha + \beta + \gamma)\, d(x_{2n}, x_{2n+1}) + (\beta + \gamma + \delta) d(x_{2n+1}, x_{2n+2})$$

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{(\alpha+\beta+\gamma)}{1-(\beta+\gamma+\delta)}\, d(x_{2n}, x_{2n+1})$$
$$d(x_{2n+1}, x_{2n+2}) \leq Cd(x_{2n}, x_{2n+1})$$

Where, $C \leq \frac{(\alpha+\beta+\gamma)}{1-(\beta+\gamma+\delta)} < 1$ … (3.1.2)

Similarly, it can be shown that,

$$d(x_{2n+3}, x_{2n+2}) \leq Cd(x_{2n+2}, x_{2n+1})$$

Therefore, for all n

$$d(x_{n+1}, x_{n+2}) \leq Cd(x_n, x_{n+1}) \leq \cdots \leq C^{n+1} d(x_0, x_1)$$

Now, for any $m > n$

$$d(x_m, x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m)$$
$$\leq [C^n + C^{n+1} + \cdots + C^{m-1}]\, d(x_0, x_1)$$
$$\leq \frac{C^n[1 - C^{m-n}]}{1 - C}\, d(x_0, x_1)$$
$$\leq \frac{C^n}{1-C}\, d(x_0, x_1)$$

From definition (2.1) we have

$$\|d(x_m, x_n)\| \leq K \frac{C^n}{1-C} \|d(x_0, x_1)\| \quad \ldots (3.1.3)$$

Which implies that $d(x_m, x_n) \to 0$ as $n, m \to \infty$.

Hence $\{x_n\}$ is a Cauchy sequence.

Since X is complete, $\exists$ a p in X such that $x_n \to p$ as $n \to \infty$ now using (3.1.1)

$$d(p, gp) \leq d(p, x_{2n+1}) + d(x_{2n+1}, gp)$$
$$\leq d(p, x_{2n+1}) + d(fx_{2n}, gp)$$
$$\leq d(p, x_{2n+1}) + \alpha d(x_{2n+1}, p) + \beta[d(x_{2n}, x_{2n+1}) + d(p, gp)] + \gamma[d(x_{2n}, gp) + d(p, x_{2n+1})] + \delta \frac{d(p,gp)[1+d(x_{2n}, x_{2n+1})]}{1+d(x_{2n},p)}$$
$$\leq d(p, x_{2n+1}) + \alpha d(x_{2n+1}, p) + \beta[d(x_{2n}, x_{2n+1}) + d(p, gp)] + \gamma[d(x_{2n}, p) + d(p, gp) + d(p, x_{2n+1})] + \delta \frac{d(p,gp)[1+d(x_{2n}, x_{2n+1})]}{1+d(x_{2n},p)}$$
$$\leq d(p, gp) \leq \frac{1}{1-\beta-\gamma-\delta}[d(p, x_{2n+1}) + \alpha d(x_{2n+1}, p) + \beta d(x_{2n}, x_{2n+1}) + \gamma[d(x_{2n}, p) + d(p, x_{2n+1})] + \delta \frac{1+d(x_{2n}, x_{2n+1})}{1+d(x_{2n},p)}]$$

From definition (2.1)

$$\|d(p, gp)\| \leq K \frac{1}{1-\beta-\gamma-\delta}\{\|d(p, x_{n+1})\| + \alpha\|d(x_{n+1}, p)\| + \beta\|d(x_n, x_{n+1})\| + \gamma\|d(x_n, p)\| + \gamma\|d(p, x_{n+1})\| + \delta \frac{1+\|d(x_n, x_{n+1})\|}{1+\|d(x_n,p)\|}\} \quad \ldots (3.1.4)$$

Now right-hand side of the above inequality approaches to 0 as $n \to \infty$.

Hence $\|d(p, gp)\| = 0$, and $p = gp$. To prove uniqueness, we have from (3.1.1)

$$d(fp, p) = d(fp, gp)$$
$$\leq \alpha d(p, p) + \beta[d(p, fp) + d(p, gp)] + \gamma[d(p, gp) + d(p, fp)] + \delta \frac{d(p,gp)[1+d(p,fp)]}{1+d(p,p)}$$

$$d(fp,p) \le (\beta+\gamma)\, d(p,fp)$$

Which, using definition of partial ordering on E and properties of cone P, give $d(fp,p)=0$, and $fp=p$. To prove uniqueness, suppose that if q is another common fixed point of f and g, then

$$d(p,q) = d(fp,gp)$$
$$\le \alpha d(p,q) + \beta[d(p,fp)+d(q,gq)]$$
$$+\ \gamma[d(p,gq)+d(q,fp)] + \ \delta\,\frac{d(q,gq)[1+d(p,fp)]}{1+d(p,q)}$$
$$d(p,q) \le (\alpha+2\gamma)d(p,q)$$

Which gives $d(p,q)=0$ and $q=p$.

Hence f and g have a unique common fixed point in X.

Corollary 3.2 Let (X,d) be a complete cone metric space, and P be a normal cone with normal constant K. Suppose that a self-map f of X satisfies:

$$d(f^p x, f^q y) \le \ \alpha d(x,y) + \beta[d(x,f^p x)+d(y,f^q y)]$$
$$+\gamma[d(x,f^q y)+d(y,f^p x)] + \delta\,\frac{d(y,f^q y)[1+d(x,f^p x)]}{1+d(x,y)} \quad \dots (3.2.1)$$

for all $x,y \in X$, where $\alpha,\beta,\gamma,\delta \ge 0, \alpha+2\beta+2\gamma+\delta<1$, and p and q are fixed positive integers. Then f has a unique fixed point in X.

Proof. Inequality (3.2.1) is obtained from (3.1.1) by setting $f \equiv f^p$ and $g \equiv f^q$.

Corollary 3.3 Let (X,d) be a complete cone metric space, and P be a normal cone with normal constant K. Suppose that mapping $f: X \longrightarrow X$ satisfies

$$d(fx,fy) \le \alpha d(x,y) + \beta[d(x,fx)+d(y,fy)] + \gamma[d(x,fy)+d(y,fx)]$$
$$+\ \delta\,\frac{d(y,fy)[1+d(x,fx)]}{1+d(x,y)} \quad \dots (3.3.1)$$

for all $x,y \in X$, where $\alpha,\beta,\gamma,\delta \ge 0$ and $\alpha+2\beta+2\gamma+\delta<1$. Then f has a unique fixed point in X.

Proof. Set, $p=q=1$ in Corollary (3.2.1).

Corollary 3.4 Let (X,d) be a complete cone metric space, and P be a normal cone with normal constant K. Suppose that mapping $f: X \longrightarrow X$ satisfies:

$$d(fx,fy) \le \ s_1 d(x,y) + s_2 d(x,fx) + s_3 d(y,fy) + s_4 d(x,fy) +$$
$$s_5 d(y,fx) \ + s_6\,\frac{d(y,fy)[1+d(x,fx)]}{1+d(x,y)} \quad \dots (3.4.1)$$

for all $x,y \in X$, where $s_i \ge 0$ for each $i \in \{1,2,\dots,5\}$ and $\sum_{i=1}^{5} s_i < 1$. Then f has a unique fixed point in X.

Proof. In inequality (3.4.1) interchanging the roles of x and y, and adding the new inequality to (3.4.1) yields (3.3.1) with $\alpha = s_1, \beta = \frac{s_2+s_3}{2}, \gamma = \frac{s_4+s_5}{2}, \delta = s_6$.

Corollary 3.5 Let (X,d) be a complete cone metric space, and P be a normal cone with normal constant K. Suppose mapping $f: X \longrightarrow X$ satisfies:

$$d(fx,fy) \le \alpha d(x,y) + \delta\left[\frac{d(y,fy)[1+d(x,fx)]}{1+d(x,y)}\right] \quad \dots (3.5.1)$$

for all $x,y \in X$, where, $\alpha,\delta \ge 0$ and $\alpha+\delta<1$. Then f has a unique fixed point in X.

Corollary 3.6 Let (X,d) be a complete cone metric space, and P be a normal cone with normal constant K. Suppose that mapping $f: X \longrightarrow X$ satisfies:

$$d(fx, fy) \leq \beta[d(x, fx) + d(y, fy)] + \gamma[d(x, fy) + d(y, fx)] \quad \ldots (3.6.1)$$

for all $x, y \in X$, where $\beta, \gamma \geq 0$ and $\beta + \gamma < \frac{1}{2}$. Then f has a unique fixed point in X.

Corollary 3.7 Let (X, d) be a complete cone metric space, and P be a normal cone with normal constant K. Suppose mapping $f: X \longrightarrow X$ satisfies:

$$d(fx, fy) \leq \alpha d(x, y) + \gamma[d(x, fy) + d(y, fx)] \quad \ldots (3.7.1)$$

for all $x, y \in X$, where $\alpha, \gamma \geq 0$ and $\alpha + 2\gamma < 1$. Then f has a unique fixed point in X.

4. References:

1. Abbas, M., & Jungck, G., Common fixed point results for non-commuting mappings without continuity in cone metric spaces. Journal of Mathematical Analysis and Applications, (2008) 341(1), 416–420.
2. Abbas, M., & Rhoades, B. E**.,** Fixed and periodic point results in cone metric spaces. Applied Mathematics Letters, (2009), 22(4), 511–51.
3. Chatterjea, S. K. Fixed-point theorems. Czechoslovak Mathematical Journal, (1971), 21(96), 55–59.
4. Ciric L. B., A generalization of Banach contraction principle, Proceedings of the American Mathematical Society, 45, 1974, 267-273.
5. Ciric, L.B., Generalized contractions and fixed-point theorems, Publ. Inst. Math. (Beograd) (N.S.), **12**(26) (1971), 19-26.
6. Haung, Zhang X., Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007) 1468–1476.
7. Jungck, G., Commuting mappings and fixed point, Amer. Math. Monthly, 83(4) (1976), 261-263.
8. Jungck, G., Compatible mappings and common fixed points, Int. J. Math. Math. Sci.,**9**(4) (1986), 771-779.
9. Kannan R., Some results on fixed points, Bull. Calcutta Math. Soc. 60 (1968) 71–76.
10. Pant, R. P., & Rakocevic, V. Fixed point and periodic point theorems. Acta Scientiarum Mathematicarum, 2024, volume 90, 175–192.
11. R. Simeon, Zaslavski A. J. A Fixed-Point Result in Generalized metric spaces. The Journal of Analysis, 30, 1467-1473, (2022)
12. Saluja A.S., Jhade P.K. and Manro S. A Fixed Point Theorem on A Class Generalized Metric Spaces, Indian Journal of Mathematics and Mathematical Science, 6, 193-197, (2010).

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RESOLVING BANACH'S CONTRACTION PRINCIPLE BY USING JAVA PROGRAMMING TO DETERMINE FIXED POINTS

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Abstract

In this paper we present a computer-based JAVA program. The program proposing a Java based implementation to find a fixed point of a linear equations or functions satisfying Banach's Contraction principle. This program approximates iteratively to the fixed point using a contraction mapping on real numbers.

Keywords: Fixed point, Metric Space, JAVA.

2020 Mathematics Subject Classification: Primary: 54H25; Secondary: 54E50,47H10.

1. Introduction

Fixed point theory plays an essential role in numerous mathematical and computational areas. In mathematical analysis, fixed point theory is a dignified and influencing branch or mathematics. This theory majorly used in differential equations, numerical methods and optimization.

In 1922 S. Banach has proved a very useful and significant result which is Banach's contraction principle.

Banach's Contraction principle States that "if f is a contraction mapping on complete metric space, then f has a unique fixed point."

Stefen Banach was a leading figure in function analysis. Fields like Topology, differential equations, numerical analysis are the areas in which his work influenced very effectively. His Contraction principle was one of the innovative results in the metric fixed point theory in mathematical analysis and has application in multiple scientific. Mathematical and computational field.

In 1912 Luitzen Brower has proved that any continuous function mapping a convex compact subset of Rn to itself has a fixed point. This work is done earlier than Banach's Contraction principle.

In 1930 Julius Schauder expanded Brower's fixed point theorem to infinite dimensional spaces. Many remarkable works have done in the field of fixed point theory. S. Kakutani developed a generalization in 1941 which was useful in game theory and economics.

In this paper we are presenting a computer program and a computational approach to find a fixed point of a linear equation satisfying Banach's Contraction principle.

Banach's Contraction principle -Let (X, d) be a complete metric space and let $f: X \rightarrow X$ be a contraction mapping, there exists constant c i.e.

$0 \leq c < 1$ such that $d(f(x), f(y)) = c\, d(x, y)$ for every $x, y \in X$

This theorem guarantees that f has a unique fixed point x which can be iteratively approximates as $x_{n+1} = f(x)$, When x_n converges to x as $n \to \infty$.

In this program we will take an initial guess x_0, and will apply the function iteratively. This program will do the process and will show that what is an exact fixed point of the given linear equation. This program will show fixed point correct up to 4 decimal places. And this program will also show that how many iterations has done to find exact fixed point.

2. **Preliminaries**

Definition 2.1. (Metric Space) [05]: A metric space is a set X equipped with a [distance function d that satisfies the following properties:

1. **Non-negativity:** For all $x, y \in X, d(x, y) \geq 0$.
2. **Identity of indiscernible:** For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.
3. **Symmetry:** For all $x, y \in X, d(x, y) = d(y, x)$.
4. **Triangle inequality:** For all $x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z)$.

The function d is called a metric on X. The pair (X, d) is called a metric space.

Definition 2.2. (Fixed Point) [04]: A point which remains invariant under the transformation is said to be a Fixed Point.
Example - Let $f: R \to R$ given by $f(x) = x^2 + 1$. The fixed points of f are the solutions to the equation $x^2 + 1 = x$; it follows that $x = 1/2 \pm \sqrt{3}/2$ are the fixed points of f.

Definition 2.3. (Contraction mapping) [01]: $Let\ X = (X, d)$ be a metric space. A mapping $T: X \to X$ is called a contraction on X if there is a positive real number $a <$ 1st for all x. y ∈ X. $d\,(Tx.Ty) \leq a\, d(x\,, y)\,,\ (a < 1)$.
Geometrically this means that any points x and y have images that are together than those points x and y; more precisely, the ratio $d\,(Tx, Ty)/d(x, y)$ does not exceed a constant a which is strictly less than 1

Definition 2.4. (Class) [02]: A class is a blueprint or template for creating objects in object-oriented programming. It defines attributes (fields/variables) and behaviours (methods/functions) that objects instantiated from the class will have.

Definition 2.5. (Method) [03]: A method in Java is a block of code that performs a specific task and can be called to execute when needed. It enhances code reusability and modularity. A method typically consists of a name, return type, parameters (optional), and a body containing executable statements.

Definition 2.6. (Datatype) [10]: A data type is a classification that specifies which type of value a variable can hold in a programming language. It determines the possible values for that type, the operations that can be performed on it, and the way the values are stored in memory.

Definition 2.7. (Variable) [03]: A variable in Java is a named memory location used to store data that can be changed during program execution. Each variable has a data type, which defines the kind of values it can hold, such as integers, floating-point numbers, or characters.

Definition 2.8. (While loop) [03]: A while loop in Java is a control flow statement that repeatedly executes a block of code as long as a specified Boolean condition evaluates to true. It is useful when the number of iterations is not known beforehand.

Definition 2.9. ((Fnction)Math.abs) [03]: The Math.abs() function in Java returns the absolute value of a given number. It removes any negative sign, ensuring the result is always non-negative. This function is overloaded to work with different numeric types such as int, long, float, and double.

Definition 2.10. (object) [03]: An object in Java is an instance of a class that encapsulates both state (attributes/fields) and behaviour (methods/functions). Objects are created using the new keyword and allow interaction with class-defined functionalities.

3. Main Approach (Java implementation)

```
public class Contraction
{
    public static double contFun(double x)
{
       return 0.4 * x + 1;
}
 public static double findFixedPoint(double x0, double e, int maxIterations)
{
       double currentX = x0;
       double nextX = contFun(currentX);
       int iteration = 1;
 System.out.printf("Iteration %d: x = %.6f%n", iteration, nextX);
 while (Math.abs(nextX - currentX) > e && iteration < maxIterations)
        {
          currentX = nextX;
           nextX = contFun(currentX);
           iteration++;
           System.out.printf("Iteration %d: x = %.6f%n", iteration, nextX);
       }
return nextX;
   }
public static void main(String[] args)
{
       double x0 = 0.0;
       //x=initial guess
       double e= 0.0001;
```

```
        //e=tolerance
        int maxIterations = 50;
 System.out.println("Finding fixed point using Banach's Contraction Principle...");
        double fixedPoint = findFixedPoint(x0, e, maxIterations);
        System.out.printf("%nApproximate Fixed Point: %.6f%n", fixedPoint);
    }
}
```

Output –

Finding fixed point using Banach's Contraction Principle...
Iteration 1: x = 1.000000
Iteration 2: x = 1.400000
Iteration 3: x = 1.560000
Iteration 4: x = 1.624000
Iteration 5: x = 1.649600
Iteration 6: x = 1.659840
Iteration 7: x = 1.663936
Iteration 8: x = 1.665574
Iteration 9: x = 1.666230
Iteration 10: x = 1.666492
Iteration 11: x = 1.666597
Iteration 12: x = 1.666639
Approximate Fixed Point: 1.666639

4. Conclusion

This paper is introducing a java based program to find the exact fixed point of a linear equation satisfying Banach's contraction principle, implementation of this program may help to provide a fixed point of various types of linear equations.

5. References

1. Banach, S., Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundamenta Mathematicae, 3(1), 133–181, (1922).
2. Booch, G., Rumbaugh, J., & Jacobson, I., The Unified Modeling Language User Guide (2nd ed.). Addison-Wesley, (2005).
3. Gosling, J., Joy, B., Steele, G., & Bracha, G., The Java Language Specification (3rd ed.). Addison-Wesley, (2005).
4. Granas, A., & Dugundji, J., Fixed Point Theory. Springer, (2003).
5. Munkres, J. R., Topology (2nd ed.). Prentice Hall, (2000).
6. Rudin, W., Real and complex analysis. McGraw-Hill, (1987).
7. R.Walter., Principles of Mathematical Analysis. McGraw-Hill, (1976).
8. Sadovskii, B.N., "The fixed point principle for multivalued mappings." Funktsional'nyi Analiz i Ego Prilozheniya, 1(2), pp. 74–76, (1967).
9. Schauder, J., "Der Fixpunktsatz in Funktionalräumen." Studia Mathematica, Vol. 2, pp. 171–180, (1930).
10. Sebesta, R. W., Concepts of Programming Languages (12th ed.). Pearson, (2018).

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A FIXED POINT RESULT ON A CLASS OF GENERALIZED METRIC SPACES

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Abstract

This paper gives a generalized proof of a fixed-point theorem for a contraction mapping in generalized metric space.

Keywords: Generalized metric space, T-Orbitally complete.
2020 Mathematics Subject Classification: Primary: 54H25; Secondary: 54E50,47H10.

1. Introduction

The concept of a generalized metric space introduced by Branciari [1] in which the triangular inequality of a metric space has been replaced by a more general inequality in which instead of three points it involving four points. As such, any metric space is generalized metric space but the converse is not true [1]. He presented the well-known Banach's fixed point theorem in such a space. a further generalization of that result has been obtained.

It becomes natural to explore whether alternative well-known fixed-point theorems—beyond those that rely strictly on contraction-type mappings—can also be established within the framework of generalized metric spaces. In this paper, we pursue this line of inquiry by directing our attention to such theorems and presenting a fixed-point result that extends existing findings through slight but meaningful modifications, specifically tailored to these broader spaces. In this paper we generalized the result of [10] by taking more general inequality.

2. Preliminaries

Following definitions are required in the sequel.

Throughout, the letters R and N will denote the set of all non-negative real numbers and the set of all positive integers respectively.

Definition 2.1: Let X be a set and $d: X^2 \rightarrow R^+$ a mapping such that for all x, y∈X and there exist a point conditions. X. different from x and y, one has following

1. $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y. x)$
3. $d(x, y) \leq d(x. z) + d(z. y)$

Then we will say that (X, d) is a metric space.

Definition 2.2: Let X be a set and $d: X^2 \rightarrow R^+$ a mapping such that for all $x, y \epsilon X$ and for all distinct point $x_1, x_2, x_3, \dots\dots, x_n \epsilon X$. cach of them different from x and y. one has

1 $d(x, y) = 0$ if and only if $x = y$
2 $d(x, y) = d(y. x)$
3 . $d(x, y) \leq d(x. x_1) + d(x_1, x_2) + \ ... + d(x_{n-1}, x_n) + d(x_n, y)$ Then we will say that $(X. d)$ is generalized metric space (or shortly g.m.s.)

Definition 2.3: 1.et $(X. d)$ be a g.m.s. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for any $\omega > 0$ there exist $n \in N$ such that for all $m, n \epsilon N$, $n \leq m$, one has $d(x_n, x_{n+m}) < \varepsilon$. Then $(X. d)$ is called complete if every Cauchy sequence in X is convergent in X.

Let T:X→ X be a mapping where X is a g. m. s. for each x ∈ X

$O(x, \infty) = \{x, Tx, T^2x \dots.\}$

Definition 2.4: X is said to be T-orbitally complete if and only if every Cauchy sequence which is contained in $O(x, \infty)$ for some $x \epsilon X$ converges in X.

3. Main Result

Theorem 3.1 Let (X, d) be a metric space, if $T: X \to X$ be a mapping such that

$$d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty) + d(x, Ty) + d(x, y)] \qquad \dots (3.1.1)$$

holds for all $x, y \epsilon X$ where $0 < \beta < \frac{1}{5}$

and if X is T-orbitally complete then T has a unique fixed point in X.

Proof: Let $x \epsilon X$ now using (1) with $y = Tx$

$$d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty) + d(x, Ty) + d(x, y)]$$
$$d(Tx, T^2x) \leq \beta[d(x, Tx) + d(Tx, T^2x) + d(x, T^2x) + d(x, Tx)]$$
$$d(Tx, T^2x)(1 - \beta) \leq \beta[2d(x, Tx) + d(x, T^2x)]$$
$$d(Tx, T^2x)(1 - \beta) \leq \beta[2d(x, Tx) + d(x, Tx) + (Tx, T^2x)]$$
$$d(Tx, T^2x)(1 - 2\beta) \leq 3\beta d(x, Tx)$$
$$d(Tx, T^2x)(1 - 2\beta) \leq \frac{3\beta}{(1-2\beta)} \ d(x, Tx) \qquad \dots(3.1.2)$$

Again, by using (3.1.1) with and

$$d(T^2x, T^3x) \leq \beta[d(Tx, T^2x) + d(T^2x, T^3x) + d(Tx, T^3x) + d(Tx, T^2x)]$$
$$d(T^2x, T^3x)(1 - \beta) \leq \beta[2d(Tx, T^2x) + d(Tx, T^3x)]$$
$$d(T^2x, T^3x)(1 - \beta) \leq \beta[2d(Tx, T^2x)) + d(Tx, T^2x) + (T^2x, T^3x)]$$
$$d(T^2x, T^3x)(1 - 2\beta) \leq 3\beta d(Tx, T^2x)$$
$$d(T^2x, T^3x) \leq \frac{3\beta}{(1-2\beta)} \ d(Tx, T^2x)$$

By (3.1.2)

$$d(T^2x, T^3x) \leq \frac{3\beta}{(1-2\beta)} \frac{3\beta}{(1-2\beta)} \ d(x, Tx)$$
$$d(T^2x, T^3x) \leq \left(\frac{3\beta}{1-2\beta}\right)^2 d(x, Tx)$$

Inductively we have

$$d(T^2x, T^3x) \leq \left(\frac{3\beta}{1-2\beta}\right)^n d(x, Tx)$$

Let $\frac{3\beta}{1-2\beta} = r$, where r < 1 therefore

$$d(T^n x, T^{n+1}x) \leq r^n d(x, Tx) \qquad \dots(3.1.3)$$

Now we can claim that $\{T^n x\}$ is Cauchy sequence.
For $m > n$ we have by the definition ()

$$d(T^n x, T^m x) \leq d(T^n x, T^{n+1} x) + d(T^{n+1} x, T^{n+2} x) + \ ... + d(T^{m-1} x, T^m x)$$

By (3.1.3)

$$d(T^n x, T^m x) \leq r^n d(x, Tx) + r^{n+1} d(x, Tx) + \ .... + r^{m-1} d(x, Tx)$$
$$d(T^n x, T^m x) \leq d(x, Tx)[r^n + r^{n+1} + \ ... + r^{m-1}]$$

Let $m = n + p, p > 1$,

$$d(T^n x, T^m x) \leq d(x, Tx)[r^n + r^{n+1} + \ ... + r^{n+p-1}]$$
$$d(T^n x, T^m x) \leq d(x, Tx)[1 + r + r^2 + \ ... + r^{p-1}]$$
$$d(T^n x, T^m x) \leq d(x, Tx) r^n \left(\frac{1-r^p}{1-r}\right)$$
$$d(T^n x, T^m x) \leq d(x, Tx) \left(\frac{r^n}{1-r}\right)$$

For all, $n \epsilon N$ since $0 < r < 1$ then $r^n \to 0$ as $n \to \infty$ and so $\{T^n x\}$ is Cauchy sequence. Since X is T-orbitally complete let $T^n x = u$ .Therefore all its subsequence also converges to u.

Now by using definition (2.2)

$$d(u, Tu) \leq d(u, T^n x) + d(T^n x, T^{n+1} x) + \ ... + d(T^{2n-1} x, T^{2n} x) + d(T^{2n} x, Tu)$$

By using (3.1.1)

$$d(u, Tu) \leq d(u, T^n x) + d(T^n x, T^{n+1} x) + \ ... + d(T^{2n-1} x, T^{2n} x) + \ \beta[d(T^{2n-1} x, T^{2n} x) + d(u, Tu) + \ d(T^{n-1} x, u)]$$
$$d(u, Tu)(1 - \beta) \leq d(u, T^n x) + d(T^n x, T^{n+1} x) + \ ... + (1 - \beta) d(T^{2n-1} x, T^{2n} x) + \beta d(T^{n-1} x, u)$$
$$d(u, Tu) \leq \frac{1}{1-\beta} d(u, T^n x) + d(T^n x, T^{n+1} x) + \ ... + (1 - \beta) d(T^{2n-1} x, T^{2n} x) + \beta d(T^{n-1} x, u)$$

Limiting $n \to \infty$

$$d(u, Tu) \leq \frac{1}{1 - \beta} [d(u, u) + d(u, u) + \ ... + (1 + \beta) d(u, u) + \beta d(u, u)]$$

$d(u, Tu) \leq 0$ implies $Tu = u$ $\quad \therefore u\ is\ fixed\ point\ off\ T.$

Uniqueness: Let v be another fixed point of T

$$\therefore Tv = v$$

Now using (3.1.1) with $x = u$ and $y = v$

$$d(Tu, Tv) \leq \beta[d(u, Tu) + d(v, Tv) + d(u, Tv) + d(u, v)]$$
$$d(u, v) \leq \beta[d(u, u) + d(v, v) + d(u, v) + d(u, v)]$$
$$d(u, v) \leq 2\beta d(u, v)$$
$$d(u, v)(1 - 2\beta) \leq 0$$

Since $0 < \beta < \frac{1}{5}$ therefore $d(u, v) = 0$ implies $u = v$

This completes the proof of theorem.

Corollary 3.2. Let (X, d) be a metric space, if $T: X \to X$ be a mapping such that

$$d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty) + d(x, Ty)]$$

holds for all $x, y \epsilon X$ where $0 < \beta < \frac{1}{4}$
and if X is T-orbitally complete then T has a unique fixed point in X.

References:

[1] Branciari A., A Fixed-Point Theorem of Banach-Caccioppoli Type on a Class of Generalized Metric Spaces, *Publ. Math. Debrecen*., 57(1-2), 31-37, (2000).
[2] C. Karim, Mohammad Dahman, Mustafa Kabil & Abdessamed Kamouss, Various Extensions of the Kannan Fixed Point Theorem in Algebra Valed Generalized Metric Spaces., Advances in Fixed Point Theory, 14, (2024).
[3] C.M. Pacurar, Fixed Point Theorem for Generalized Chatterjea Type Mappings., Acta Matheatica Hungarica, 173, 500-509, (2024).
[4] Ciric L.j.B., On Contraction Type Mappings, *Math. Balkanica,* 1, 52-57, (1971).
[5] Ciric L.j.B. A Generalization of Banach's Contraction Principal, *Proc. Amer. Math. Soc.*, 45(2), 267-273, (1974).
[6] Jleli, M., & Samet, B., A generalized metric space and related fixed-point theorems. Fixed Point Theory and Algorithms for Science and Engineering, (1), 2, (2015).
[7] Kannan. R., Some results on fixed points. American Mathematical Monthly, 76(4), 405-408, (1969).
[8] R. Simeon & Alexander J. Zaslavski, A Fixed-Point Result in Generalized metric spaces, The Journal of Analysis, 30, 1467-1473, (2022).
[9] S. Mustapha, A Fixed-Point Result in Generalized Metric Space with a graph., Advances in Fixed Point Theory, 13, (2023).
[10] Saluja A.S. & Mukesh Kumar Jain, A Fixed-Point Theorem on A Class Generalized Metric Spaces, Indian Journal of Mathematics and Mathematical Science, 6, 193-197, (2010).

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