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*This Volume of*  
**INSPIRE**  
*is being dedicated to*  
***Varahamihira: Master Astronomer and Mathematician***

Varaha or Mihir, was an Indian astronomer, Indian astronomer, mathenatician, and astrologer and who lived in Ujjain. He was born in Avanti region, roughly corresponding to modern-day Malwa, to Adityadasa, who was himself an astronomer. According to one of his own works, he was educated at Kapitthaka. He is considered to be one of the nine jewels (Navarata (Navaratnas)) of the court of legendary ruler Yoshdharman Vikramaditya of Malwa.

He was the first one to mention in his work Pancasiddhantika that the avanamsa, or the shifting of the equinox, is 50.32 seconds.

Varahamihira's main work is the book Pancasiddhantika (or Pancha-Siddhantika, "[Treatise] on the Five [Astronomical] (Canons) dated ca. 575 CE gives us information about older Indian texts which are now lost. The work is a treatise on mathematical astronomy and it summarises five earlier astronomical treatises, namely the Surya Siddhanta, Romaka siddhanta, Paulisa Siddhanta, Vasihtha Siddhanta and Paitamaha Siddhantas . It is a compendium of Vedanga Jyotisha as well as Hellenistic astronomy (including Greek, Egyptian and Roman elements). He was the first one to mention in his work Pancha Siddhantika that the ayanamsa, or the shifting of the equinox is 50.32 seconds.

## *FOREWORD*

The present volume of *INSPIRE* contains the various research papers of Faculty and Research Scholars of Department of Mathematics, INSTITUTE FOR EXCELLENCE IN HIGHER EDUCATION, BHOPAL (M. P.).

For me it is the realization of a dream which some of us have been nurturing for long and has now taken a concrete shape through the frantic efforts and good wishes of our dedicated band of research workers in our country, in the important area of mathematics.

The editor deserves to be congratulated for this very successful venture. The subject matter has been nicely and systematically presented and is expected to be of use to the workers.

(Dr. M. L. Nath)  
Director & Patron  
IEHE, Bhopal (M. P.)

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## FRACTIONAL-CALCULUS RESULTS PERTAINING TO I-FUNCTION

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### ABSTRACT

Fractional Calculus and generalized hypergeometric functions have contributed a lot to the theory of science and engineering. In view of importance and usefulness of fractional-calculus operators in different directions, we present a number of key results for the product of two  $I$ -functions involving the Riemann-Liouville, the Weyl and such other fractional-calculus operators as those based upon the Cauchy-Goursat Integral Formula. The results discussed here can be used to investigate a wide class of new and known results.

### 1. INTRODUCTION:

Fractional Calculus is the field of applied mathematics that deals with the derivatives and the integrals of arbitrary orders. During the last three decades Fractional Calculus has been applied to almost every field of science, engineering and mathematics. Many applications of Fractional Calculus can be found in fluid dynamics, Stochastic dynamical system, plasma physics and controlled thermonuclear fusion, image processing, nonlinear control theory, nonlinear biological system, astrophysics, etc.

Two of the most commonly encountered tools in the theory and applications of Fractional Calculus are provided by the Riemann-Liouville operator  $R_z^\nu$  ( $\nu \in \mathbb{C}$ ) and the Weyl  $W_z^\nu$  ( $\nu \in \mathbb{C}$ ) operator which are defined by [1, see also [3, 4, 8]]. In this paper, we will define these operators as follows:

$$R_z^\nu \{f(z)\} = \begin{cases} \frac{1}{\Gamma(\nu)} \int_0^z (z-\xi)^{\nu-1} f(\xi) d\xi & (Re(\nu) > 0) \\ \frac{d^n}{dz^n} R_z^{\nu+n} \{f(z)\} & (-n < Re(\nu) \leq 0; n \in \mathbb{N}) \end{cases} \quad (1.1)$$

and

$$W_z^\nu \{f(z)\} = \begin{cases} \frac{1}{\Gamma(\nu)} \int_z^\infty (\xi-z)^{\nu-1} f(\xi) d\xi & (Re(\nu) > 0) \\ \frac{d^n}{dz^n} W_z^{\nu+n} \{f(z)\} & (-n < Re(\nu) \leq 0; n \in \mathbb{N}) \end{cases} \quad (1.2)$$

provided that the defining integrals exist,  $N$  being the set of positive integers. The following definition of a fractional differintegral of order  $\nu \in \mathbb{C}$  is based essentially upon the familiar Cauchy-Goursat Integral Formula:

**Definition** (Cf. [2 and 9]) If the function  $f(z)$  is analytic (regular) inside and on  $C$ , where

$$C = \{C^-, C^+\}, \quad (1.3)$$

$C^-$  is a contour along the cut joining the points  $z$  and  $-\infty + i\Im(z)$ , which starts from the point at  $-\infty$ , encircles the point  $z$  once contour clockwise and returns to the point at  $-\infty$ ,  $C^+$  is a contour along the cut joining the points  $z$  and  $\infty + i\Im(z)$ , which starts from the point at  $\infty$ , encircles the point  $z$  once contour clockwise and returns to the point at  $\infty$ .

$$f_\nu(z) = (f(z))_\nu = \frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{f(\xi)}{(\xi-z)^{\nu+1}} d\xi, \quad (1.4)$$

$$(\nu \in \mathbb{C} \setminus \mathbb{Z}^-; \mathbb{Z}^- = \{-1, -2, -3, \dots\})$$

and

$$f_{-n}(z) = \lim_{\nu \rightarrow -n} \{f_\nu(z)\}, \quad (n \in N = \{1, 2, 3, \dots\}) \quad (1.5)$$

where  $\xi \neq z$ ,

$$-\pi \leq \arg(\xi - z) \leq \pi \text{ for } C^-, \quad (1.6)$$

and

$$0 \leq \arg(\xi - z) \leq 2\pi \text{ for } C^+, \quad (1.7)$$

Then  $f_\nu(z)$  ( $\operatorname{Re}(\nu) > 0$ ) is said to be the fractional derivative of  $f(z)$  of order  $\nu$  and  $f_\nu(z)$  ( $\operatorname{Re}(\nu) < 0$ ) is said to be the fractional integral of  $f(z)$  of order  $-\nu$ , provided that

$$|f_\nu(z)| < \infty \quad (\nu \in \mathbb{R}). \quad (1.8)$$

The  $I$ -function, defined by Saxena [5], has been further studied by other workers [7 and 10]. In this paper we will define and represent the  $I$ -function in the following manner:

$$I[x] = I_{P_i, Q_i, R}^{M, N} \left[ x \left| \begin{matrix} [(c_j, \gamma_j)_{1, N}, (c_{ji}, \gamma_{ji})_{N+1, P_i}] \\ [(d_j, \delta_j)_{1, M}, (d_{ji}, \delta_{ji})_{M+1, Q_i}] \end{matrix} \right. \right] = \frac{1}{2\pi\omega} \int_{\mathcal{L}} \theta(s) x^s ds, \quad (1.9)$$

where

$$\theta(s) = \frac{\prod_{j=1}^M \Gamma(d_j - \delta_j s) \prod_{j=1}^N \Gamma(1 - c_j + \gamma_j s)}{\sum_{i=1}^R \left\{ \prod_{j=M+1}^{q_i} \Gamma(1 - d_{ji} + \delta_{ji} s) \prod_{j=N+1}^{p_i} \Gamma(c_{ji} - \gamma_{ji} s) \right\}}, \quad (1.10)$$

where  $\mathcal{L}$  is a suitable contour,  $\omega = \sqrt{-1}$  and all other conditions given in literature [6].

By summing up the residues at the simple poles of the integrand of (1.9), the following expression is obtained:

$$I[x] = \sum_{k=1}^M \sum_{h=0}^{\infty} \left\{ \frac{(-1)^h \Theta(\zeta_{k,h})}{(h)! \delta_k} x^{\zeta_{k,h}} \right\}, \quad (1.11)$$

where

$$\zeta_{k,h} = \frac{d_k + h}{\delta_k}, \quad h = 0, 1, 2, \dots \quad (1.12)$$

and

$$\Theta(\zeta_{k,h}) = \frac{\theta(t)}{\Gamma(d_k - \delta_k t)}, \quad (1.13)$$

provided that the series on the right hand side of (1.3) is absolutely convergent.

## 2. THE MAIN FRATIONAL DIFFERINTEGRAL FORMULAS:

First of all, for the Riemann-Liouville operator  $R_Z^\nu$  defined by (1.1), we have

$$\begin{aligned} & R_Z^\nu \{ z^{\rho-1} I_{p_i, q_i, r}^{m, n}(xz^\sigma) I_{P_i, Q_i, R}^{M, N}(yz^\tau) \} \\ &= \frac{1}{\Gamma(\nu)} \int_0^z (z - \xi)^{\nu-1} \xi^{\rho-1} I_{p_i, q_i, r}^{m, n}(x\xi^\sigma) I_{P_i, Q_i, R}^{M, N}(y\xi^\tau) d\xi, \quad (Re(\nu) > 0). \end{aligned} \quad (2.1)$$

Now, expressing the one  $I$ -function in series form as given by (1.11) and another  $I$ -function in terms of Melline-Barnes type of contour integral given by (1.9), interchanging the orders of summation and integration and putting  $\xi = zt$  in the resulting integral, we find that

$$\begin{aligned} & R_Z^\nu \{ z^{\rho-1} I_{p_i, q_i, r}^{m, n}(xz^\sigma) I_{P_i, Q_i, R}^{M, N}(yz^\tau) \} = \frac{z^{\nu+\rho-1}}{\Gamma(\nu)} \sum_{k=1}^M \sum_{h=0}^{\infty} \left\{ \frac{(-1)^h \Theta(\zeta_{k,h})}{(h)! \delta_k} x^{\zeta_{k,h}} z^{\sigma \zeta_{k,h}} \right\} \\ & \cdot \frac{1}{2\pi\omega} \int_{\mathcal{L}} \theta(s) (yz^\tau)^s ds \int_0^1 t^{\rho+\sigma \zeta_{k,h} + \tau s - 1} (1-t)^{\nu-1} dt, \quad (Re(\nu) > 0). \end{aligned} \quad (2.2)$$

Further, we evaluate the Eulerian integral in (2.2) by applying the following integral representation for the familiar Beta function  $B(\alpha, \beta)$ :

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt = \int_0^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}} dt = B(\beta, \alpha)$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} , \quad \{Re(\alpha) > 0, Re(\beta) > 0\}. \quad (2.3)$$

and interpret the resulting integral in (2.2) as the  $I$ -function by means of the definition (1.9), we get

$$\begin{aligned} & R_z^v \{ z^{\rho-1} I_{p_i, q_i, r}^{m, n}(xz^\sigma) I_{P_i, Q_i, R}^{M, N}(yz^\tau) \} \\ &= z^{v+\rho-1} \sum_{k=1}^M \sum_{h=0}^{\infty} \left\{ \frac{(-1)^h \Theta(\zeta_{k,h})}{(h)! \delta_k} x^{\zeta_{k,h}} z^{\sigma \zeta_{k,h}} \right\} \\ & \bullet I_{P_i+1, Q_i+1, R}^{M, N+1} \left[ yz^\tau \left| \begin{array}{l} (1 - \rho - \sigma \zeta_{k,h}, \tau), (c_j, \gamma_j)_{1, N}, (c_{ji}, \gamma_{ji})_{N+1, P_i} \\ (d_j, \delta_j)_{1, M}, (d_{ji}, \delta_{ji})_{M+1, Q_i}, (1 - \rho - \sigma \zeta_{k,h} - \nu, \tau) \end{array} \right. \right], \quad (2.4) \\ & \left( \tau > 0, \sigma > 0; Re(\nu) > 0, Re(\rho + \sigma \zeta_{k,h}) + \tau \min_{1 \leq j \leq m} \left\{ Re \left( \frac{d_j}{\delta_j} \right) \right\} > 0 \right), \end{aligned}$$

provided that the series on the right hand side of (2.4) is absolutely convergent.

In precisely the same manner as mentioned above, by applying the definition (1.2) with

$$\xi = z(1+t) \text{ and } d\xi = z dt,$$

and evaluating the resulting infinite integral as a Beta function by means of (2.3), we find that

$$\begin{aligned} & W_z^v \{ z^{-\rho} I_{p_i, q_i, r}^{m, n}(xz^{-\sigma}) I_{P_i, Q_i, R}^{M, N}(yz^{-\tau}) \} \\ &= z^{v-\rho} \sum_{k=1}^M \sum_{h=0}^{\infty} \left\{ \frac{(-1)^h \Theta(\zeta_{k,h})}{(h)! \delta_k} x^{\zeta_{k,h}} z^{-\sigma \zeta_{k,h}} \right\} \\ & \bullet I_{P_i+1, Q_i+1, R}^{M, N+1} \left[ yz^{-\tau} \left| \begin{array}{l} (1 + \nu - \rho - \sigma \zeta_{k,h}, \tau), (c_j, \gamma_j)_{1, N}, (c_{ji}, \gamma_{ji})_{N+1, P_i} \\ (d_j, \delta_j)_{1, M}, (d_{ji}, \delta_{ji})_{M+1, Q_i}, (1 - \rho - \sigma \zeta_{k,h}, \tau) \end{array} \right. \right], \quad (2.5) \\ & \left( \tau > 0, \sigma > 0; Re(\nu) > 0, Re(\rho + \sigma \zeta_{k,h}) + \tau \min_{1 \leq j \leq m} \left\{ Re \left( \frac{c_j - 1}{\gamma_j} \right) \right\} > 0 \right), \end{aligned}$$

provided that each members of (2.5) exist.

Next, we make use of the definition (1.4) for which it is known that [1, Vol. I, p. 28]

$$\begin{aligned} (z^k)_\nu &= e^{-i\pi\nu} \frac{\Gamma(\nu - k)}{\Gamma(-k)} z^{k-\nu}, \quad (2.6) \\ &\left( k, \nu \in \mathbb{C}; \left| \frac{\Gamma(\nu - k)}{\Gamma(-k)} \right| < \infty \right). \end{aligned}$$

In this case, too, we choose to apply the series representation for the first  $I$ -function and for the second  $I$ -function in terms of Mellin-Barnes type of contour integral given by (1.9), and we thus obtain

$$\begin{aligned} & \left( z^{-\rho} I_{p_i, q_i, r}^{m, n}(xz^{-\sigma}) I_{P_i, Q_i, R}^{M, N}(yz^{-\tau}) \right) \\ &= z^{-\nu-\rho} e^{-i\pi\nu} \sum_{k=1}^M \sum_{h=0}^{\infty} \left\{ \frac{(-1)^h \Theta(\zeta_{k,h})}{(h)! \delta_k} x^{\zeta_{k,h}} z^{-\sigma\zeta_{k,h}} \right\} \\ & \bullet I_{P_i+1, Q_i+1, R}^{M, N+1} \left[ yz^{-\tau} \left| \begin{matrix} (1-\nu-\rho-\sigma\zeta_{k,h}, \tau), (c_j, \gamma_j)_{1,N}, (c_{ji}, \gamma_{ji})_{N+1, P_i} \\ (d_j, \delta_j)_{1,M}, (d_{ji}, \delta_{ji})_{M+1, Q_i}, (1-\rho-\sigma\zeta_{k,h}, \tau) \end{matrix} \right. \right], \quad (2.7) \\ & \quad (\tau > 0, \sigma > 0), \end{aligned}$$

provided that each members of (2.7) exist.

### 3. SPECIAL CASES:

- (i) Taking  $x = 0, R = 1$  in (2.4), we arrive at the following results involving Fox's  $H$ -function

$$\begin{aligned} & R_z^\nu \{ z^{\rho-1} H_{P,Q}^{M,N}(yz^\tau) \} \\ &= z^{\nu+\rho-1} H_{P+1, Q+1}^{M, N+1} \left[ yz^\tau \left| \begin{matrix} (1-\rho, \tau), (c_j, \gamma_j)_{1,P} \\ (d_j, \delta_j)_{1,Q}, (1-\rho-\nu, \tau) \end{matrix} \right. \right], \quad (3.1) \\ & \quad \left( \tau > 0, Re(\nu) > 0, Re(\rho) + \tau \min_{1 \leq j \leq m} \left\{ Re \left( \frac{d_j}{\delta_j} \right) \right\} > 0 \right), \end{aligned}$$

provided that each members of (3.1) exist.

- (ii) Taking  $x = 0, R = 1$  in (2.5), we arrive at the following results involving Fox's  $H$ -function

$$\begin{aligned} & W_z^\nu \{ z^{-\rho} H_{P,Q}^{M,N}(yz^{-\tau}) \} \\ &= z^{\nu-\rho} H_{P+1, Q+1}^{M, N+1} \left[ yz^{-\tau} \left| \begin{matrix} (1+\nu-\rho, \tau), (c_j, \gamma_j)_{1,P} \\ (d_j, \delta_j)_{1,Q}, (1-\rho, \tau) \end{matrix} \right. \right], \quad (3.2) \\ & \quad \left( \tau > 0, Re(\rho) + \tau \min_{1 \leq j \leq m} \left\{ Re \left( \frac{c_j - 1}{\gamma_j} \right) \right\} > Re(\nu) > 0 \right), \end{aligned}$$

provided that each members of (3.2) exist.

- (iii) Taking  $x = 0, R = 1$  in (2.7), we arrive at the following results involving Fox's  $H$ -function

$$\begin{aligned} & \left( z^{-\rho} H_{P,Q}^{M,N}(yz^{-\tau}) \right)_v \\ &= z^{-v-\rho} e^{-i\pi v} H_{P+1,Q+1}^{M,N+1} \left[ yz^{-\tau} \left| \begin{matrix} (1-v-\rho, \tau), (c_j, \gamma_j)_{1,P} \\ (d_j, \delta_j)_{1,Q}, (1-\rho, \tau) \end{matrix} \right. \right], \quad (3.3) \\ & \quad (\tau > 0, \sigma > 0), \end{aligned}$$

provided that each members of (3.3) exist.

#### 4. CONCLUSION:

The importance of our result lies in their manifold generality. In view of the generality of the  $I$ -function, on specializing the various parameters, we can obtain from our results, several results involving a remarkable wide variety of useful functions, which are expressible in terms of Fox's  $H$ -function, Meijer's  $G$ -function etc. and their special cases. Thus, the result established in this paper would at once yield a very large number of results involving a large variety of special functions occurring in the problem of science, engineering and mathematics.

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## ON COMMON FIXED POINT THEOREMS

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### ABSTRACT

In this paper we establish a common fixed point theorem for a quadruple of self mappings in a normed space using Mann Iteration.

**Keywords:** Common fixed points, Mann iteration.

**2000 MSC:** 54H25, 47H10.

### 1. INTRODUCTION:

Throughout this paper  $T_1$  &  $T_2$  be two self mappings of a Banach space  $B$ . The Mann iteration process associated with  $T_1$  &  $T_2$  is defined in the following manner:

Let  $x_0 \in N$  and let

$$x_{2n+1} = (1 - c_{2n})x_{2n} + c_{2n}T_1x_{2n}$$

$$x_{2n+2} = (1 - c_{2n+1})x_{2n+1} + c_{2n+1}T_2x_{2n+1}$$

For  $n = 0, 1, 2, \dots$  where  $c_n$  satisfies,

- (i)  $c_0 = 1$  (ii)  $0 < c_n < 1$ ,  $n = 1, 2, \dots$  & (iii)  $\lim_{n \rightarrow \infty} c_n = h > 0$

Pathak [1] prove the following theorems for a quadruple of self mappings as follows:

**Theorem 1.1:** Let  $X$  be a closed convex subset of a normed space  $N$ . Let  $T_1$  &  $T_2$  be mappings of  $X$  into  $X$  and let  $f$  &  $g$  be injective and continuous mappings of  $X$  into  $X$  satisfying:

$$\|T_1x - T_2y\| \leq q \max \left\{ \|fx - gy\|, \frac{\|fx - T_1x\| [1 - \|fx - T_2y\|]}{1 + \|fx - T_1x\|}, \frac{\|fx - T_2y\| [1 - \|fx - T_1x\|]}{1 + \|fx - T_2y\|}, \frac{\|T_1x - gy\| [1 - \|gy - T_2y\|]}{1 + \|T_1x - gy\|}, \frac{\|gy - T_2y\| [1 - \|T_1x - gy\|]}{1 + \|gy - T_2y\|} \right\} \quad (1.1)$$

$$\|fx - fgy\| \leq \|T_1x - fx\| + \|T_1x - T_2y\| + \|T_2y - gy\| + \|gy - fx\| \quad (1.2)$$

$$\|gy - gfx\| \leq \|T_1x - gy\| + \|T_1x - T_2y\| + \|T_2y - fx\| + \|gy - fx\| \quad (1.3)$$

for all  $x, y$  in  $X$ , where  $0 < q < 1$ ,

$$(1 - \lambda) f(X) + \lambda T_1(X) \subseteq g(X) \quad (1.4)$$

$$(1 - \mu) g(X) + \mu T_2(X) \subseteq f(X) \quad (1.5)$$

for all  $\lambda, \mu \in (0, 1]$ , the sequence  $\{x_n\}$  associated with the mapping  $T_1, T_2, f$  &  $g$  is defined by

$$x_{2n+1} \in g^{-1}[(1 - c_{2n})fx_{2n} + c_{2n}T_1x_{2n}] \quad (1.6)$$

$$x_{2n+2} \in f^{-1}[(1 - c_{2n+1})gx_{2n+1} + c_{2n+1}T_2x_{2n+1}] \quad (1.7)$$

$n=0, 1, 2, \dots$

Where  $x_0$  is an arbitrary point in  $X$  and  $\{y_n\}$  is the sequence defined by  $y_{2n-1} = fx_{2n-1}$  &  $y_{2n} = gx_{2n}$  for  $n=1, 2, \dots$  and  $\{c_n\}$  satisfies condition (i), (ii), and (iii) given above. If  $\{y_n\}$  converges to a point  $u$  in  $X$ , then  $u$  is the unique common fixed point of  $T_1, T_2, f$  &  $g$ .

In our present paper we use another contractive condition in place of (1.1).

## 2. MAIN RESULT:

**Theorem 2.1:** Let  $X$  be a closed convex subset of a normed space  $N$ . Let  $T_1, T_2$  be mappings of  $X$  into  $X$  and let  $f$  &  $g$  be injective and continuous mappings of  $X$  into  $X$  satisfying:

$$\|T_1x - T_2y\| \leq q \max \left\{ \frac{\|fx - T_1x\|[\|gy - T_2y\| + \|T_1x - gy\|]}{\|fx - gy\| + \|T_1x - gy\|}, \right. \\ \frac{\|T_1x - gy\|[\|fx - T_1x\| + \|gy - T_2y\|]}{\|fx - gy\| + \|T_1x - gy\|}, \\ \frac{\|gy - T_2y\|[1 + \|fx - T_1x\|]}{1 + \|fx - gy\|}, \\ \left. \frac{\|fx - T_1x\|\|gy - T_2y\|}{\|fx - gy\|}, \|fx - gy\| \right\} \quad (2.1)$$

with (1.2), (1.3), (1.4), and (1.5). Also the sequence  $\{x_n\}$  associated with the mappings  $T_1, T_2, f$  &  $g$  is defined by (1.6) and (1.7) and  $\{y_n\}$  is the sequence defined by  $y_{2n-1} = fx_{2n-1}$  &  $y_{2n} = gx_{2n}$  for  $n=1, 2, \dots$  and  $\{c_n\}$  satisfies condition (i), (ii), and (iii) given above. If  $\{y_n\}$  converges to a point  $u$  in  $X$ , then  $u$  is the unique common fixed point of  $T_1, T_2, f$  &  $g$ .

**Proof:** We observe that since  $f$  and  $g$  are injective and satisfy condition (1.4) and (1.5) the sequence  $\{x_n\}$  defined by the equations (1.6) and (1.7) is unique.

Also from (1.6)

$$x_{2n+1} \in g^{-1}[(1-c_{2n})fx_{2n} + c_{2n}T_1x_{2n}] \\ \Rightarrow T_1x_{2n} = \frac{gx_{2n+1} - (1-c_{2n})fx_{2n}}{c_{2n}}$$

Taking limit  $n \rightarrow \infty$ , we get

$$\lim_{n \rightarrow \infty} T_1x_{2n} = \lim_{n \rightarrow \infty} \frac{gx_{2n+1} - (1-c_{2n})fx_{2n}}{c_{2n}} \\ = \frac{u - (1-h)u}{h} = u$$

(as sequence  $\{y_n\}$  converges to a point  $u$  in  $X$  so  $\{y_{2n}\} \rightarrow u$  &  $\{y_{2n+1}\} \rightarrow u$  and by condition (iii)).

Similarly from (1.7), we get  $\lim_{n \rightarrow \infty} T_2x_{2n+1} = u$

From equation (1.2), we have

$$\|fx_{2n} - gx_{2n+1}\| \leq \|T_1x_{2n} - fx_{2n}\| + \|T_1x_{2n} - T_2x_{2n+1}\| + \\ \|T_2x_{2n+1} - gx_{2n+1}\| + \|gx_{2n+1} - fx_{2n}\|$$

And so

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \|fx_{2n} - fgx_{2n+1}\| &\leq \lim_{n \rightarrow \infty} \{\|T_1x_{2n} - fx_{2n}\| + \|T_1x_{2n} - T_2x_{2n+1}\| + \\
 &\quad \|T_2x_{2n+1} - gx_{2n+1}\| + \|gx_{2n+1} - fx_{2n}\|\} \\
 \Rightarrow \lim_{n \rightarrow \infty} \|y_{2n} - fy_{2n+1}\| &\leq \lim_{n \rightarrow \infty} \{\|T_1x_{2n} - y_{2n}\| + \|T_1x_{2n} - T_2x_{2n+1}\| + \\
 &\quad \|T_2x_{2n+1} - y_{2n+1}\| + \|y_{2n+1} - y_{2n}\|\} \\
 \Rightarrow \|u - fu\| &\leq \|u - u\| + \|u - u\| + \|u - u\| + \|u - u\| \\
 \Rightarrow \|u - fu\| &\leq 0 \\
 \Rightarrow u &= fu
 \end{aligned}$$

Also from (1.3)

$$\begin{aligned}
 \|gx_{2n+1} - gfx_{2n}\| &\leq \|T_1x_{2n} - gx_{2n+1}\| + \|T_1x_{2n} - T_2x_{2n+1}\| + \\
 &\quad \|T_2x_{2n+1} - fx_{2n}\| + \|gx_{2n+1} - fx_{2n+1}\| \\
 \Rightarrow \lim_{n \rightarrow \infty} \|gx_{2n+1} - gfx_{2n}\| &= \lim_{n \rightarrow \infty} \|y_{2n+1} - gy_{2n}\| = \|u - gu\| \leq 0
 \end{aligned}$$

It follows that  $u = gu$ .

Further using inequality (2.1), we have

$$\begin{aligned}
 \|u - T_2u\| &\leq \|u - gx_{2n+1}\| + \|gx_{2n+1} - T_2u\| \\
 &\leq \|u - gx_{2n+1}\| + \|(1 - c_{2n})fx_{2n} + c_{2n}T_1x_{2n} - T_2u\| \\
 &\leq \|u - gx_{2n+1}\| + (1 - c_{2n})\|fx_{2n} - T_2u\| + c_{2n}\|T_1x_{2n} - T_2u\|
 \end{aligned}$$

$$\begin{aligned}
 \|u - T_2u\| &\leq \|u - gx_{2n+1}\| + (1 - c_{2n})\|fx_{2n} - T_2u\| + \\
 q \max &\left\{ \frac{\|fx_{2n} - T_1x_{2n}\|[\|gu - T_2u\| + \|T_1x_{2n} - gu\|]}{\|fx_{2n} - gu\| + \|T_1x_{2n} - gu\|}, \right. \\
 &\quad \frac{\|T_1x_{2n} - gu\|[\|fx_{2n} - T_1x_{2n}\| + \|gu - T_2u\|]}{\|fx_{2n} - gu\| + \|T_1x_{2n} - gu\|}, \\
 &\quad \frac{\|gu - T_2u\|[1 + \|fx_{2n} - T_1x_{2n}\|]}{1 + \|fx_{2n} - gu\|}, \\
 &\quad \left. \frac{\|fx_{2n} - T_1x_{2n}\|\|gu - T_2u\|}{\|fx_{2n} - gu\|}, \|fx_{2n} - gu\| \right\}
 \end{aligned}$$

Assume that  $T_2u \neq u$ , we have on letting  $n \rightarrow \infty$ ,

$$\begin{aligned}\|u - T_2u\| &\leq 0 + (1-h)\|u - T_2u\| + hq \max\{0, 0, \|u - T_2u\|, 0, 0\} \\ &\leq (1-h)\|u - T_2u\| + hq\|u - T_2u\| \\ &\leq (1-h+hq)\|u - T_2u\|\end{aligned}$$

a contradiction , and so  $T_2u = u$  .

Similarly,

$$\begin{aligned}\|u - T_1u\| &\leq \|u - fx_{2n+2}\| + \|fx_{2n+2} - T_1u\| \\ &\leq \|u - fx_{2n+2}\| + \|(1-c_{2n+1})gx_{2n+1} + c_{2n+1}T_2x_{2n+1} - T_1u\| \\ &\leq \|u - fx_{2n+2}\| + (1-c_{2n+1})\|gx_{2n+1} - T_1u\| + c_{2n+1}\|T_1u - T_2x_{2n+1}\| \\ &\leq \|u - fx_{2n+2}\| + (1-c_{2n+1})\|gx_{2n+1} - T_1u\| + \\ &\quad c_{2n+1}q \max\left\{\frac{\|fu - T_1u\|[\|gx_{2n+1} - T_2x_{2n+1}\| + \|T_1u - gx_{2n+1}\|]}{\|fu - gx_{2n+1}\| + \|T_1u - gx_{2n+1}\|},\right. \\ &\quad \frac{\|T_1u - gx_{2n+1}\|[\|fu - T_1u\| + \|gx_{2n+1} - T_2x_{2n+1}\|]}{\|fu - gx_{2n+1}\| + \|T_1u - gx_{2n+1}\|}, \\ &\quad \frac{\|gx_{2n+1} - T_2x_{2n+1}\|[1 + \|fu - T_1u\|]}{1 + \|fu - gx_{2n+1}\|}, \\ &\quad \left.\frac{\|fu - T_1u\|\|gx_{2n+1} - T_2x_{2n+1}\|}{\|fu - gx_{2n+1}\|}, \|fu - gx_{2n+1}\|\right\}\end{aligned}$$

Assuming that  $T_1u \neq u$  , we have on letting  $n \rightarrow \infty$  ,

$$\begin{aligned}\|u - T_1u\| &\leq (1-h)\|u - T_1u\| + hq \max\{0, 0, 0, 0, 0\} \\ \Rightarrow \|u - T_1u\| &\leq (1-h)\|u - T_1u\|\end{aligned}$$

a contradiction , hence  $T_1u = u$  .

We have therefore proved that  $u$  is a common fixed point of  $T_1, T_2, f$  &  $g$  .

To prove uniqueness: Suppose that  $v$  be another fixed point of  $T_1, T_2, f$  &  $g$  . Then,

$$\begin{aligned}
 \|u - v\| &= \|T_1 u - T_2 v\| \\
 &\leq q \max \left\{ \frac{\|fu - T_1 u\| [\|gv - T_2 v\| + \|T_1 u - gv\|]}{\|fu - gv\| + \|T_1 u - gv\|}, \right. \\
 &\quad \frac{\|T_1 u - gv\| [\|fu - T_1 u\| + \|gv - T_2 v\|]}{\|fu - gv\| + \|T_1 u - gv\|}, \\
 &\quad \frac{\|gv - T_2 v\| [1 + \|fu - T_1 u\|]}{1 + \|fu - gv\|}, \\
 &\quad \left. \frac{\|fu - T_1 u\| \|gv - T_2 v\|}{\|fu - gv\|}, \|fu - gv\| \right\} \\
 \Rightarrow \|u - v\| &\leq q \max \left\{ \frac{\|u - u\| [\|v - v\| + \|u - v\|]}{\|u - v\| + \|u - v\|}, \right. \\
 &\quad \frac{\|u - v\| [\|u - u\| + \|v - v\|]}{\|u - v\| + \|u - v\|}, \\
 &\quad \left. \frac{\|v - v\| [1 + \|u - u\|]}{1 + \|u - v\|}, \frac{\|u - u\| \|v - v\|}{\|u - v\|}, \|u - v\| \right\}
 \end{aligned}$$

$$\Rightarrow \|u - v\| \leq q \|u - v\|$$

$$\text{Therefore } (1 - q) \|u - v\| \leq 0 \quad (0 < q < 1)$$

$$\Rightarrow u = v$$

Hence  $u$  is a common unique fixed point of  $T_1, T_2, f$  &  $g$ .

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## SYNTHESIS AND CHARACTERIZATION OF MERCAPTOETHANOL CAPPED CDS NANOPARTICLES

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CdS Nanoparticles , Mercaptoethanol, UV -Visible absorption spectra , XRD.

### ABSTRACT

Semiconducting optoelectronic materials play functional role in variety of applications due to their extraordinary optical, electrical and magnetic properties. The modification of the optical, electrical, magnetic and physical properties of semiconductor materials is possible by varying sizes, structures and morphologies. The Cadmium Sulphide (CdS) is a II-VI Semiconductor materials with a direct band gap of 2.42 eV at room temperature with many outstanding optical properties which have promising applications in multiple technical fields including solar cells, detectors for laser and infrared , gas sensor , luminescence devices and optoelectronic devices .

The Cadmium Sulphide (CdS) nanoparticles of different sizes were synthesized by chemical method using Mercaptoethanol ( $C_2H_5OSH$ ) as a capping agent. Mercaptoethanol capped CdS quantum dots were obtained in aqueous solution by using Cadmium Chloride ( $CdCl_2$ ) as Cadmium source and Sodium Sulphide( $Na_2S$ ) as a Sulphur source. Mercaptoethanol as capping agent was used to control the size of the nanoparticles and its concentration was varied to obtain the different sizes of CdS nanoparticles.

X- Ray Diffraction (XRD) technique was used for structural characterization of nanoparticles of CdS, which verify the crystalline form. The average size of the nanocrystallites was measured by Debye-Scherrer formula and the particles size lies in the nm ranges. The optical characterization of the nanoparticles of CdS was carried out by UV- Visible absorption spectra and the optical band gap of synthesized CdS nanoparticles was match the theoretical range.

## **1. INTRODUCTION:**

Nanoparticle or an ultrafine particle is a small solid whose physical dimension lies between 1 to 100 nanometers. Nanotechnology is the coming revolution in molecular engineering, and therefore, it is curiosity-driven and promising area of technology. The field of nanoscience and nanotechnology is interdisciplinary in nature and being studied by physicists, chemists, material scientist, biologists, engineers, computer scientists, etc.

Research in the field of nanoparticles has been triggered by the recent availability of revolutionary instruments and approaches that allow the investigation of material's properties with a resolution close to the atomic level. Strongly connected to such technological advances are the pioneering studies that have revealed new physical properties of matter at a level intermediate between atomic / molecular and bulk.

CdS is an excellent material for optoelectronic applications. It is used for detection of visible light having a maximum sensitivity near 2.4 eV; which corresponds to its band gap. Absorption in bulk CdS is excitonic in nature with binding energy of exciton  $\sim 28$  meV, which is responsible for charge carrier generation. CdS a direct band gap material (II-VI) can be used in the fabrication of optoelectronic devices such as solar cell, laser, light emitting diodes, photoconductors and other optical devices based on its non-linear properties. It has wide biomolecular applications such as selective ion probes, fluorescence labeling, and targeting cancer cells. In particular, chalcogenides such as CdS have received much attention for potential applications in future optoelectronic, nanodevice [1-3] and biological labeling [4] due to the tunable electronic band gap depending on size and shape of nanocrystals [5,6].

The present paper deals with the synthesis of CdS nanoparticles using environment friendly, safe, inexpensive, facile and non-organometallic synthetic route and a systematic study of the CdS nanocrystals samples with varying concentration of capping agent Mercaptoethanol solution. The CdS nanoparticle has been characterized by XRD and UV-Visible absorption.

## **2. EXPERIMENTAL:**

### **2.1 Synthesis of CdS nanocrystals**

The Cadmium Sulphide (CdS) nanoparticles of different sizes have been synthesized by chemical method using Mercaptoethanol ( $C_2H_5OSH$ ) as a capping agent. Mercaptoethanol capped CdS quantum dots were prepared in aqueous solution by using Cadmium Chloride ( $CdCl_2$ ) as Cadmium source and Sodium Sulphide ( $Na_2S$ ) as a Sulphur source. The concentration of Mercaptoethanol capping agent was varied to obtain nanoparticles of CdS of different sizes.

The weight of various chemicals taken for preparing the solution is given in Table (1).

**Table (1): Quantities of different Chemicals**

| S. No. | Sample Name | Concentration capping agent (In ml) | Quantity of chemicals (for 0.01M in 100 ml) |                                    |
|--------|-------------|-------------------------------------|---------------------------------------------|------------------------------------|
|        |             |                                     | Cadmium Chloride (CdCl <sub>2</sub> )       | Sodium Sulphide(Na <sub>2</sub> S) |
| 1      | CdS - I     | 0.07                                | 0.1834gram/100ml                            | 0.078046gram/100ml                 |
| 2      | CdS - II    | 0.14                                | 0.1834gram/100ml                            | 0.078046gram/100ml                 |
| 3      | CdS - III   | 0.21                                | 0.1834gram/100ml                            | 0.078046gram/100ml                 |
| 4      | CdS - IV    | 0.28                                | 0.1834gram/100ml                            | 0.078046gram/100ml                 |
| 5      | CdS - V     | 0.32                                | 0.1834gram/100ml                            | 0.078046gram/100ml                 |

The aqueous solution of mercaptoethanol was added drop-wise in the solution of CdCl<sub>2</sub> with the help of burette at the rate of 1 ml per minute stirring the solution continuously by using magnetic stirrer. Thereafter, solution of Na<sub>2</sub>S was mixed drop-wise in the solution of CdCl<sub>2</sub> and mercaptoethanol with the help of burette at the rate of 1 ml per minute stirring the solution continuously by using magnetic stirrer. Subsequently yellow colour solution obtained was kept for 24 hrs. Yellow precipitate settled down in the bottom of the flask. This precipitate was removed and washed several times with the double distilled water.

## 2.2 Characterization

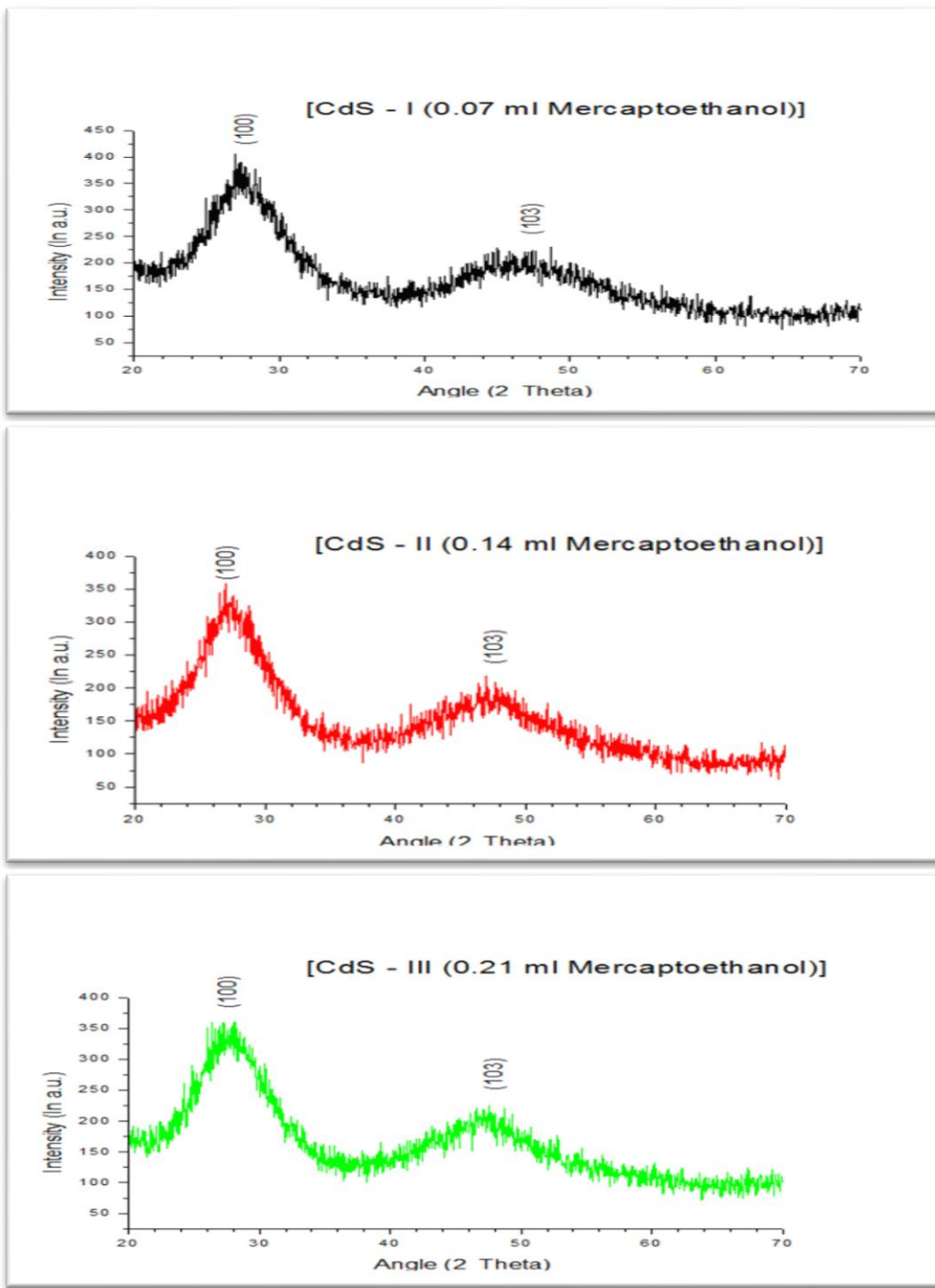
The CdS powder samples were characterized by using Bruker D8 advance X – Ray diffractometer and the optical absorption using Perkin Elemer Lambda 950, spectrometer at IUC Indore.

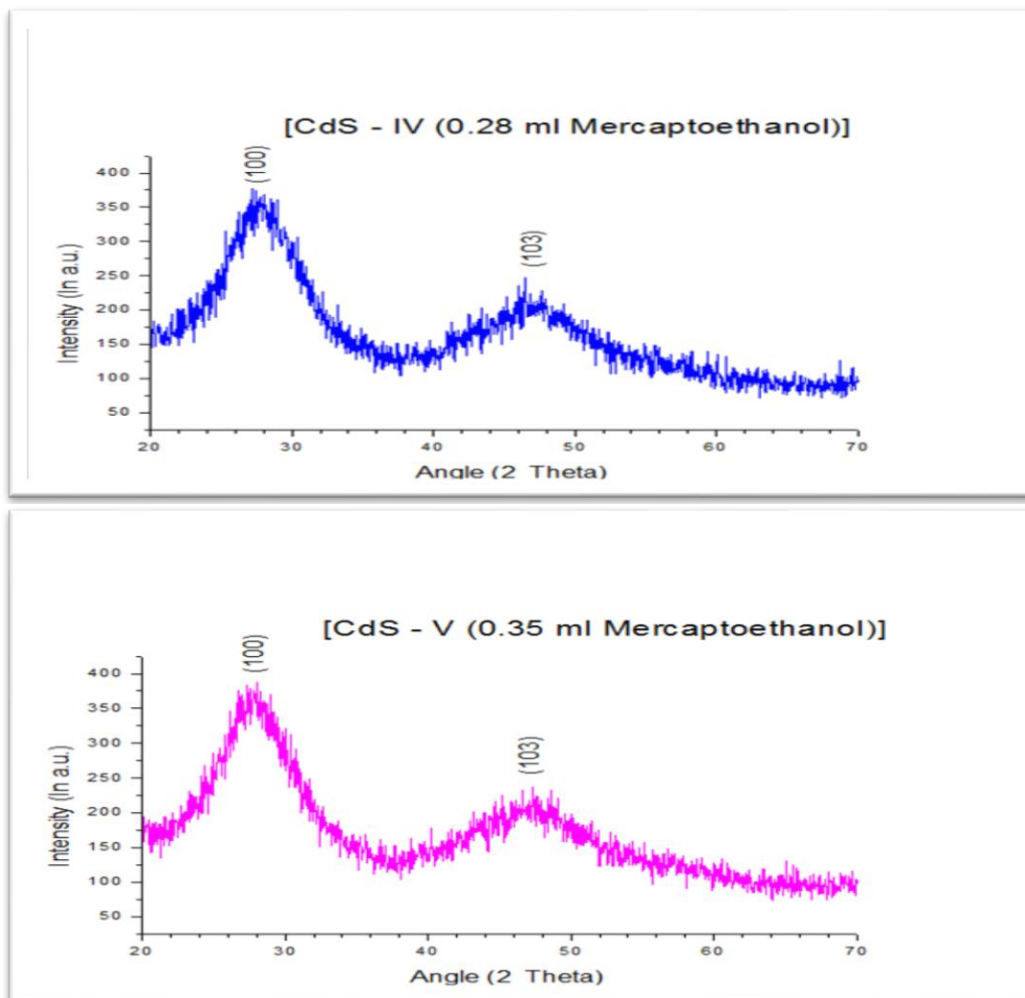
## 3. RESULTS AND DISCUSSIONS:

### 3.1 XRD structural analysis

The structural properties of CdS nanoparticles were determined by X - Ray Diffraction (XRD). The XRD patterns give information about the crystal structure. The structural characterization of CdS nanocrystals were performed by an X - ray diffractometer (Bruker D8) using CuK<sub>α</sub> radiation ( $\lambda = 1.5418 \text{ \AA}$ ) for  $2\theta$  values from  $20^\circ$  to  $70^\circ$ . X- ray diffraction pattern of CdS nanocrystals is shown in Figure (1). In the XRD of CdS nanocrystals, it appears that all CdS samples have hexagonal phase with peak at  $2\theta = 26.9473^\circ$  corresponding to (100) planes.

Due to the size effect, the XRD peaks tend to broaden and their widths become larger as the particles become smaller. As no extra peak was observed in the XRD pattern, clearly this indicated the phase purity and absence of impurity phases.





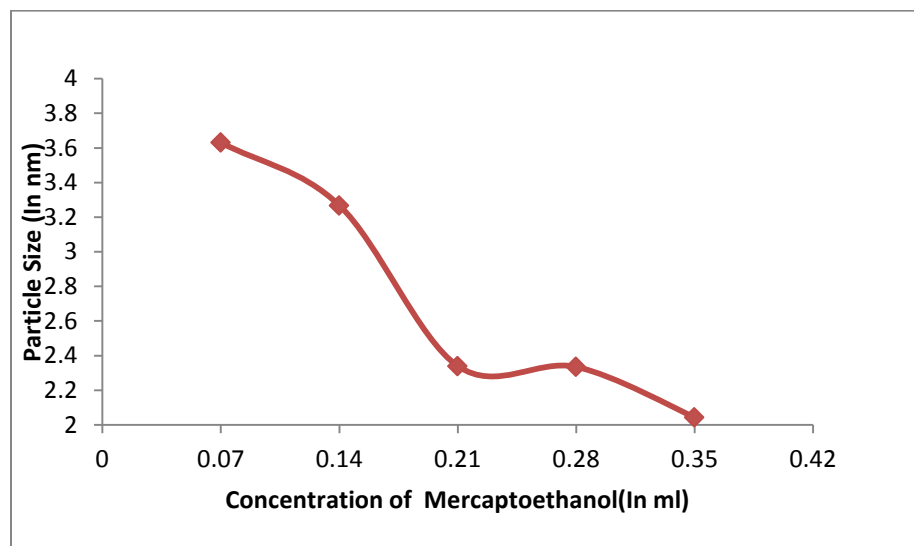
**Figure (1) : XRD pattern of CdS naocrystals of different concentration of Mercaptoethanol**

The crystallite size of the CdS have been determined from the broadening of diffraction peaks, using Debye Scherrer's formula [7],

$$D = k\lambda/\beta\cos\theta, \quad \dots\dots\dots(1)$$

where  $\lambda$  is the wavelength of X – ray used ( $\lambda = 1.54 \text{ \AA}$ ),  $\beta$  is the FWHM,  $k$  is Scherrer - constant,  $D$  is crystallite size and  $\theta$  is the Bragg's angle.

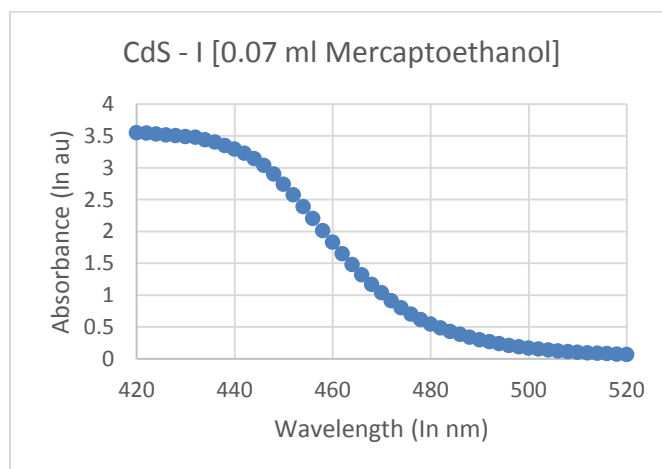
The crystalline size calculated from above formula was found 3.631 nm for CdS - I sample, as the concentration of Mercaptoethanol increased in the CdS preparation , particle size decreases and reaches to 2.044 nm for CdS - V sample. The trend of variation of crystal sizes with concentration of Mercaptoethanol are shown in Figure (2). It is seen that crystal size decreases on increasing the concentration of Mercaptoethanol capping agent.

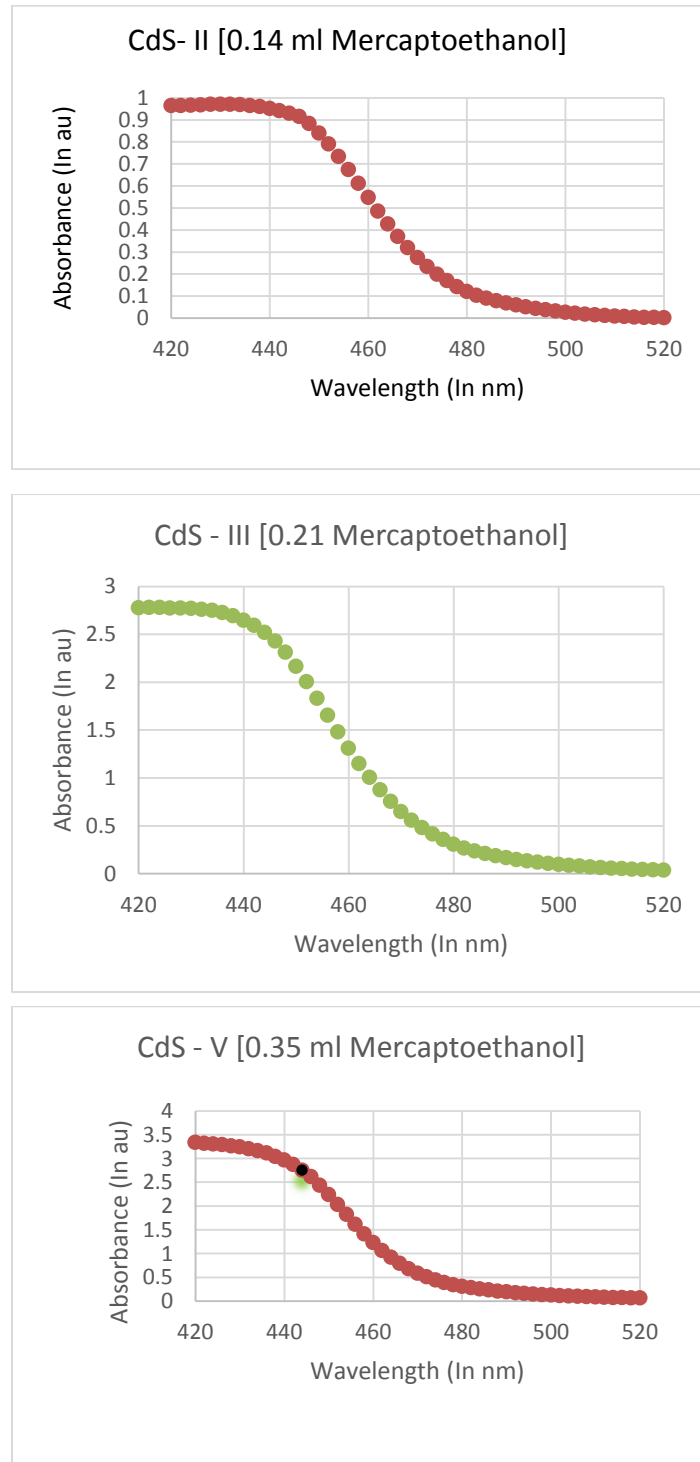


**Figure (2): Variation of Crystal Size of CdS nanocrystals with capping agent concentration**

### 3.2 UV -Visible absorption study

The optical studies of Mercaptoethanolcapped CdS nanoparticles were done by UV- Visible absorption. The optical absorption of prepared CdS nanocrystals were recorded at room temperature using LAMBDA 950 UV/Vis/NIR Spectrophotometer. The absorption spectra of CdS nanocrystals for different concentration of Mercaptoethanol taken at room temperature are shown in Figure (3). From the spectrograph it is clear that, the absorption edges of the samples are found in the range 420 - 520 nm. The optical absorption edge of the nanoparticles is shifted towards the shorter wavelength region with the increase of the concentration of capping agent Mercaptoethanol.





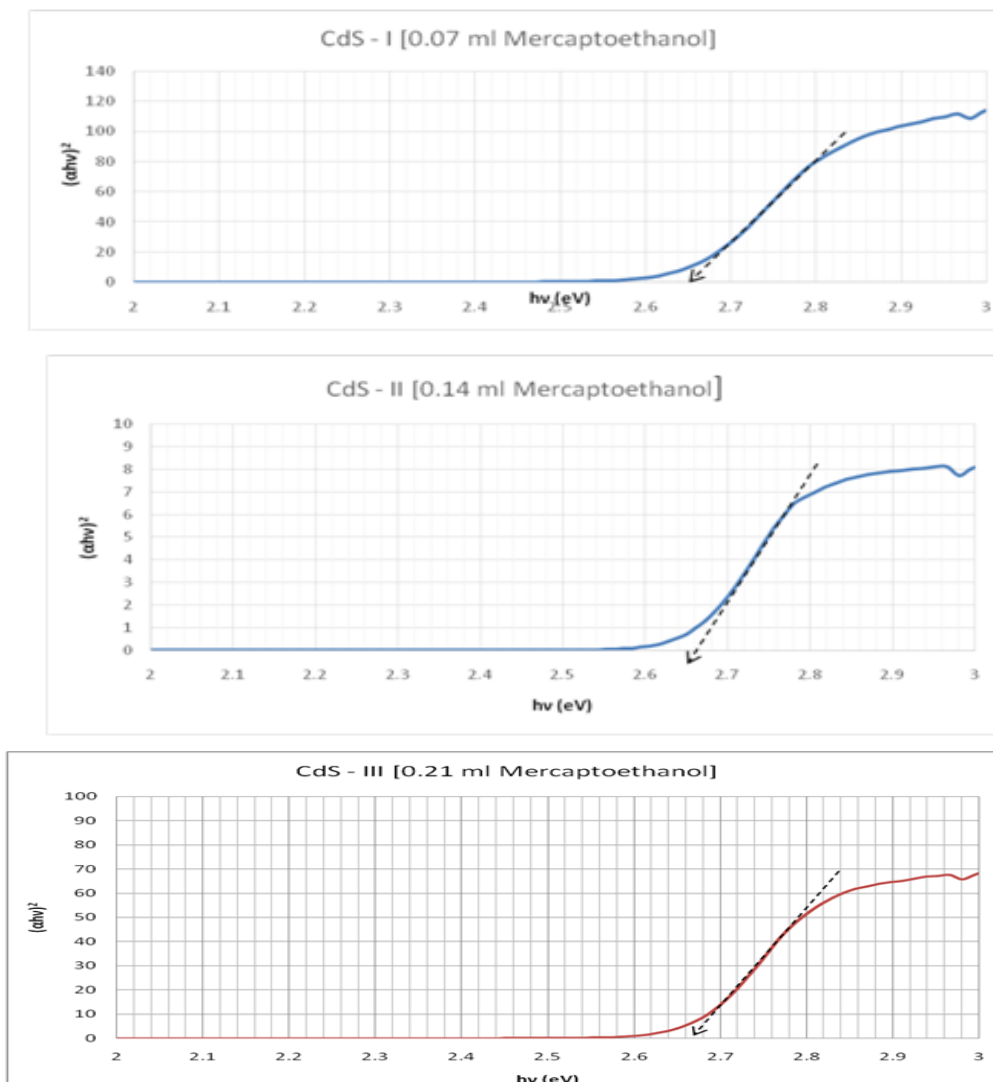
**Figure (3) : Absorption spectra of CdS nanocrystals with different concentration of Mercaptoethanol**

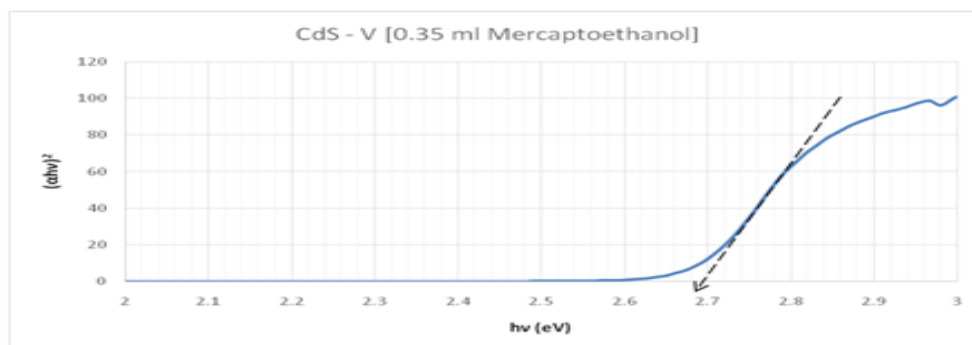
The optical band gap ( $E_g$ ) of CdS nanocrystals were determined from the absorption spectrum using the Tauc relation -

$$\alpha = \frac{A(h\nu - E_g)^n}{h\nu} \dots \dots \dots (2)$$

where  $\alpha$  is absorption coefficient ( $\text{cm}^{-1}$ ),  $h\nu$  the photon energy (eV). A and n are constants. A is complex parameter, which depends on temperature, photon energy, phonon energies etc. The n values 0.5, 1.5, 2 and 3 are for allowed direct, forbidden direct, allowed indirect and forbidden indirect transition respectively [8].  $E_g$  is the direct band gap of the CdS material.

A plot of  $(\alpha h\nu)^2$  vs  $h\nu$  should be a straight line whose intercept to the  $h\nu$  axis gives the optical band gap. The graphs  $(\alpha h\nu)^2$  vs  $h\nu$  for various concentration of Mercaptoethanol are shown in Figure (4).





**Figure (4) : Determination of band gap for CdS nanocrystals**

The experimental values of band gap are found to be larger than that the bulk material ( $E_g = 2.42$  eV) for all CdS samples with different concentration of capping agent. These are given in table (2). There is blue shift of band edge energy for CdS samples with different concentration of capping agent. This is attributed to size quantization in nanocrystals. The sizes of the nanoparticles decreases with the increasing the concentration of capping agent. The size quantization occurs due to confinement (localization) of electrons and holes in an extremely small volume of space of the nanocrystals. So, the band gap increases with the increase the concentration of capping agent or decrease of nanocrystal size.

The crystal size or nanoparticle size (diameter) of CdS nanocrystals was calculated from Effective Mass Approximation (EMA) theory by using following energy gap equation [9,10],

$$\Delta E \text{ (eV)} = \frac{h^2 \pi^2}{2r^2} \left[ \frac{1}{m_e^*} + \frac{1}{m_h^*} \right] \quad \dots\dots\dots(3)$$

$$\Delta E = E_{\text{qdot}} - E_{\text{gbulk}} \quad \dots\dots\dots(4)$$

where  $E_{\text{qdot}}$  is quantum dot energy band gap,  $E_{\text{gbulk}}$  bulk energy band gap,  $r$  is the radius of quantum dot and  $m_e^*$  ( $m_e^* = 0.19 m_e$ ) and  $m_h^*$  ( $m_h^* = 0.80 m_e$ ) values of effective mass of electrons and holes for the CdS nanocrystals.

**TABLE (2): Absorption spectra for various concentration of capping agent**

| S. no. | Sample    | Concentration capping agent (In ml) | Effective Energy Band gap $E_g$ (In eV) | Radius of particles (In nm) | Diameter of particles |
|--------|-----------|-------------------------------------|-----------------------------------------|-----------------------------|-----------------------|
| 1      | CdS - I   | 0.07                                | 2.65                                    | 3.253                       | 6.508                 |
| 2      | CdS - II  | 0.14                                | 2.66                                    | 3.185                       | 6.371                 |
| 3      | CdS - III | 0.21                                | 2.67                                    | 3.121                       | 6.242                 |
| 4      | CdS - V   | 0.35                                | 2.69                                    | 3.003                       | 6.006                 |

The effective band gap  $E_g$  of the CdS nanocrystals is obtained from the absorption spectra and the increase ( $\Delta E$ ) has been determined by subtracting band gap for bulk 2.42 eV. The particle size estimated using the above relation is given in column 6 of Table (2). It is clear from the table that the particle sizes were reduced as the capping agent concentration is increased.

#### 4. CONCLUSIONS:

The important conclusions drawn in the present investigation are follows:-

1. CdS nanoparticles have been synthesized by using Mercaptoethanol as a capping agent through a chemical synthesis route at room temperature.
2. XRD studies reveal that the CdS nanocrystals have hexagonal structure ; crystal sizes obtained by XRD are below 4 nm using Scherrer's formula.
3. From XRD analysis it is found that there crystal size decreases by increasing the concentration of capping agent Mercaptoethanol.
4. For CdS nanocrystals , there is uniform absorption in the range 200 – 800 nm, absorption edge is observed in the region of 420 – 450 nm. The absorption edge is found at lower wavelength indicating increased band gap energy as compare to bulk CdS.
5. Blue shift is observed in the absorption spectra with decreasing size of particles. This may be consequence of quantum confinement effect.
6. The Band gap of CdS nanocrystals have been obtained by absorption studies and increase in band gap has been observed on increasing the concentration of capping agent.

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## AN ATOMIC WASTE DISPOSAL PROBLEM INVOLVING I-FUNCTION

By

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### ABSTRACT

The aim of this paper is to determine the solution of a mathematical equation related to 'Atomic Waste Disposal Problem' with the help of I-function.

### 1. INTRODUCTION:

For several years the Atomic Energy Commission (now known as the Nuclear Regulatory Commission) had disposed of concentrated radioactive waste material by placing it in tightly sealed drums, which were then dumped at sea in fifty fathoms (300 feet) of water. When concerned ecologists and scientists questioned this practice, they were assured by the A.E.C. that the drums would never develop leaks. Exhaustive tests on the drums proved the A.E.C. right. However, several engineers then raised the question of whether the drums could crack from the Impact of hitting the ocean floor. "Never," said the A.E.C. "We'll see about that," said the engineers. After performing numerous experiments, the engineers found that the drums could crack on impact if their velocity exceeded forty feet per second. The problem before us, therefore, is to compute the velocity of the drums upon impact with the ocean floor. To this end, we digress briefly to study elementary Newtonian mechanics.

Newtonian mechanics is the study of Newton's famous laws of motion and their consequences. Newton's first law of motion states that an object will remain at rest, or move with constant velocity, if no force is acting on it. A force should be thought of as a push or pull. This push or pull can be exerted directly by something in contact with the object, or it can be exerted indirectly, as the earth's pull of gravity is.

Newton's second law of motion is concerned with describing the motion of an object, which is acted upon by several forces. Let  $y(t)$  denote the position of the center of gravity of the object. (We assume that the object moves in only one direction.)

Those forces acting on the object, which tend to increase  $y$ , are considered positive, while those forces tending to decrease  $y$  are considered negative. The resultant force  $F$  acting on an object is defined to be the sum of all positive forces minus the sum of all negative forces. Newton's second law of motion states that the acceleration  $d^2y/dt^2$  of an object is proportional to the resultant force  $F$  acting on it; i.e.,

$$d^2y/dt^2 = F/m \quad (1)$$

The constant  $m$  is the mass of the object. It is related to the weight  $W$  of the object by the relation  $W = mg$ , where  $g$  is the acceleration of gravity. Unless otherwise stated, we assume that the weight of an object and the acceleration of gravity are constant. We will also adopt the English system of units, so that  $t$  is measured in seconds,  $y$  is measured in feet, and  $F$  is measured in pounds. The units of  $m$  are then slugs, and the gravitational acceleration  $g$  equals  $32.2 \text{ ft/s}^2$ .

The I-function of one variable is defined by Saxena [1, p.366-375] and we will represent here in the following manner:

$$I_{P_i, Q_i; r}^{m, n} [x]_{[(a_j, \alpha_j)_{1, n}], [(a_{ji}, \alpha_{ji})_{n+1, p_i}]}^{[(b_j, \beta_j)_{1, m}], [(b_{ji}, \beta_{ji})_{m+1, q_i}]} = \frac{1}{2\pi\omega} \int_L \theta(s) x^s ds \quad (2)$$

where  $\omega = \sqrt{-1}$ ,

$$\theta(s) = \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j s) \prod_{j=1}^n \Gamma(1 - a_j - \alpha_j s)}{\sum_{i=1}^r \left[ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} + \beta_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \alpha_{ji} s) \right]}$$

integral is convergent, when  $(R > 0, S \leq 0)$ , where

$$R = \sum_{j=1}^n \alpha_j - \sum_{j=n+1}^{p_i} \alpha_{ji} + \sum_{j=1}^m \beta_j - \sum_{j=m+1}^{q_i} \beta_{ji}, \quad (3)$$

$$S = \sum_{j=1}^{p_i} \alpha_{ji} - \sum_{j=1}^{q_i} \beta_{ji},$$

$|\arg x| < \frac{1}{2} R\pi, \forall i \in (1, 2, \dots, r)$ .

## 2. MATHEMATICAL MODEL:

We return now to our atomic waste disposal problem. As a drum descends through the water, it is acted upon by three forces  $W$ ,  $B$ , and  $D$ . The force  $W$  is the weight of the drum pulling it down, and in magnitude,  $W = 527.436 \text{ lb}$ . The force  $B$  is the buoyancy force of the water acting on the drum. This force pushes the drum up, and its magnitude is the weight of the water displaced by the drum.

Now, the Atomic Energy Commission used 55 gallon drums, whose volume is  $7.35 \text{ ft}^3$ . The weight of one cubic foot of salt water is 63.99lb. Hence  $B = (63.99)(7.35) = 470.327 \text{ lb}$ .

The force  $D$  is the drag force of the water acting on the drum; it resists the motion of the drum through the water. Experiments have shown that any medium such as water, oil, and air resists the motion of an object through it. This resisting force acts in the direction opposite the motion, and is usually directly proportional to the velocity  $V$  of the object. Thus,  $D = cV$ , for some positive constant  $c$ . Notice that the drag force increases, as  $V$  increases, and decreases as  $V$  decreases. To calculate  $D$ , the engineers conducted numerous towing experiments. They concluded that the orientation of the drum had little effect on the drag force, and that  $D = 0.08 V \text{ (lb)(s)/ft}$ .

Now, set  $y = 0$  at sea level, and let the direction of increasing  $y$  be downwards. Then,  $W$  is a positive force, and  $B$  and  $D$  are negative forces. Consequently, from (1),

$$d^2y/dt^2 = (W - B - cV)/m = (g/W)(W - B - cV).$$

We can rewrite this equation as a first-order linear differential equation for  $V = dy/dt$ ; i.e.

$$dV/dt + (cg/W)V = (g/W)(W - B). \quad (4)$$

Initially, when the drum is released in the ocean, its velocity is zero. Thus,  $V(t)$ , the velocity of the drum, satisfies the initial-value problem

$$dV/dt + (cg/W)V = (g/W)(W - B), \quad V(0) = 0. \quad (5)$$

and this implies that ;

$$V(t) = [(W - B)/c] [1 - e^{(-cg/W)t}]. \quad (6)$$

Equation (6) expresses the velocity of the drum as a function of time. In order to determine the impact velocity of the drum, we must compute the time  $t$  at which the drum hits the ocean floor. Unfortunately, though, it is impossible to find  $t$  as an explicit function of  $y$ . Therefore, we cannot use Equation (6) to find the velocity of the drum when it hits the ocean floor. However, the A.E.C. can use this equation to try and prove that the drums do not crack on impact. To overcome this problem here we are giving a solution of equation (5) in terms of I-function, which can be helpful to determine the solution of the above raised problem, since I-function may be reduced to Legendre functions, Bessel functions etc.

### 3. Solution in terms of I-function:

Choose concentration  $V(t)$  in terms of I-function [1] as

$$V(t) = I_{p_i, q_i; r}^{m, n} [zt^\mu]_{[(a_j, \alpha_j)_{1, n}], [(a_{ji}, \alpha_{ji})_{n+1, p_i}]}^{[(b_j, \beta_j)_{1, m}], [(b_{ji}, \beta_{ji})_{m+1, q_i}]} \quad (7)$$

where  $\mu > 0$ ,  $|\arg z| < \frac{1}{2} \pi R$ , where  $R$  is given in (3).

Now differentiate it with respect to t, we get

$$\frac{dV(t)}{dt} = \frac{1}{t} I_{p_i+1, q_i+1; r}^{m, n+1} [zt^\mu]_{\dots\dots\dots(1, \mu)}^{(0, \mu), \dots\dots\dots} \quad (8)$$

Now after using (7) and (8) in (5), we get following result

$$\begin{aligned} & \frac{1}{t} I_{p_i+1, q_i+1; r}^{m, n+1} [zt^\mu]_{\dots\dots\dots(1, \mu)}^{(0, \mu), \dots\dots\dots} + \frac{cg}{W} I_{p_i, q_i; r}^{m, n} [zt^\mu]_{[(b_j, \beta_j)_{1, m}], [(b_{ji}, \beta_{ji})_{m+1, q_i}]}^{[(a_j, \alpha_j)_{1, n}], [(a_{ji}, \alpha_{ji})_{n+1, p_i}]} \\ & = (g/W) (W - B), \quad V(0) = 0. \end{aligned} \quad (9)$$

where  $\mu > 0$ ,  $|\arg z| < \frac{1}{2} \pi R$ , where R is given in (3)..

#### 4. SPECIAL CASES:

On specializing the parameters, I-function may be reduced to H-function, G-function, Lauricella's functions Legendre functions, Bessel functions, hypergeometric functions, Appell's functions, Kampe de Fariet's functions and several other higher transcendental functions. Therefore the result (9) is of general nature and may reduced to be in different forms, which will be useful in the literature on applied Mathematics and other branches.

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## IMPROPER INTEGRALS INVOLVING THE PRODUCTS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

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### ABSTRACT

The object of this paper is to evaluate an infinite integral involving the product of I-function, generalized hypergeometric function and confluent hypergeometric function by means of finite difference operations E. As the generalized hypergeometric function and I-function are of a very general nature, the integral, on specializing the parameters, leads to a generalization of many results some of which are known and other are believed to be new.

**(Key Words-** Generalized Hypergeometric Functions, I-function and E-operator).

### 1. INTRODUCTION:

The I-function introduced by Saxena [7], will be represented and defined as

$$I_{p_i, q_i; r}^{m, n} \left[ x \left| \begin{matrix} (a_j, \alpha_j)_{1, n}, (a_{ji}, \alpha_{ji})_{n+1, p_i} \\ (b_j, \beta_j)_{1, m}, (b_{ji}, \beta_{ji})_{m+1, q_i} \end{matrix} \right. \right] = \frac{1}{2\pi\omega} \int_L \theta(\xi) x^\xi d\xi, \quad (1.1)$$

where

$$\theta(\xi) = \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j \xi) \prod_{j=1}^n \Gamma(1 - a_j + \alpha_j \xi)}{\sum_{i=1}^r \left[ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} + \beta_{ji} \xi) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \alpha_{ji} \xi) \right]}. \quad (1.2)$$

For the nature of contour  $L$  in (1.1), the convergence, existence conditions and other details of the I-function, one can refer to [9].

### 2. FORMULAE USED:

The following formulae will be used in the present work:

From Erdélyi [3, p. 337, eq. (8)]:

$$\int_0^{\infty} x^{s-1} e^{-\alpha x} W_{\mu, \nu}(\beta x) dx = \beta^{\nu+1/2} \frac{\Gamma(\nu + s + 1/2) \Gamma(-\nu + s + 1/2)}{\Gamma(s - \mu + 1) (\alpha + \beta/2)^{\nu+s+1/2}} \cdot {}_2F_1 \left[ \left( \nu + s + 1/2, \nu - \mu + 1/2 \right); (s - \mu + 1); \frac{2\alpha - \beta}{2\alpha + \beta} \right], \quad (2.1)$$

$Re(s) > |Re(\mu)| - 1/2$ .

The finite difference operator  $E$  is [(4), p. 33, with  $w = 1$ ]:

$$E_a f(a) = f(a + 1) \quad (2.2)$$

and

$$(\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} \quad (2.3)$$

### 3. IMPROPER INTEGRALS:

The integrals to be established in this section are

$$\begin{aligned} & \int_0^{\infty} x^{s-1} e^{-1/2 \beta x} W_{\mu, \nu}(\beta x) I_{p_i, q_i, r}^{m, n} \left[ z x^k \left| \begin{matrix} (a_j, \alpha_j)_{1, n}, (a_{ji}, \alpha_{ji})_{n+1, p_i} \\ (b_j, \beta_j)_{1, m}, (b_{ji}, \beta_{ji})_{m+1, q_i} \end{matrix} \right. \right] dx \\ &= \beta^{-s} I_{p_i+2, q_i+1, r}^{m, n+2} \left[ z \beta^{-k} \left| \begin{matrix} (1/2 \pm \nu - s, k), (a_j, \alpha_j)_{1, n}, (a_{ji}, \alpha_{ji})_{n+1, p_i} \\ (b_j, \beta_j)_{1, m}, (b_{ji}, \beta_{ji})_{m+1, q_i}, (\mu - s, k) \end{matrix} \right. \right], \quad (3.1) \\ & Re(\beta) > 0, Re(s + b_j/\beta_j) > |Re(\nu)| - \frac{1}{2}; (j = 1, \dots, k). \end{aligned}$$

$$\begin{aligned} & \int_0^{\infty} x^{s-1} e^{-1/2 \beta x} W_{\mu, \nu}(\beta x) {}_uF_v(A_u; B_u; C x^d) \\ & \quad \times I_{p_i, q_i, r}^{m, n} \left[ z x^k \left| \begin{matrix} (a_j, \alpha_j)_{1, n}, (a_{ji}, \alpha_{ji})_{n+1, p_i} \\ (b_j, \beta_j)_{1, m}, (b_{ji}, \beta_{ji})_{m+1, q_i} \end{matrix} \right. \right] dx \\ &= \beta^{-s} \sum_{h=0}^{\infty} \frac{(A_v)_h C^h}{(B_v)_h h! \beta^{hd}} I_{p_i+2, q_i+1, r}^{m, n+2} \\ & \quad \times \left[ z \beta^{-k} \left| \begin{matrix} (1/2 \pm \nu - dh - s, k), (a_j, \alpha_j)_{1, n}, (a_{ji}, \alpha_{ji})_{n+1, p_i} \\ (b_j, \beta_j)_{1, m}, (b_{ji}, \beta_{ji})_{m+1, q_i}, (\mu - s - dh, k) \end{matrix} \right. \right], \quad (3.2) \end{aligned}$$

where

$$Re(s + b_j/\beta_j) > |Re(\nu)| - 1/2; (j = 1, \dots, k).$$

#### 4. PROOFS:

The integral (3.1) can be evaluated by replacing the  $I$ -function on the left hand side by its equivalent contour integral (1.1) , changing the order of integration , which is justified due to the absolute convergence of the integrals, evaluating the inner integral with the help of [3, p. 337, eq. (8) with  $\alpha = \beta/2$ ].

The integral (3.2) can be obtained on multiplying both sides of (3.1) by

$$\frac{\prod_{j=1}^u \Gamma(A_j + \delta) C^\delta}{\prod_{j=1}^v \Gamma(B_j + \delta)}$$

and applying the operator  $\exp(E_\delta^d E_\delta)$ , we get

$$\exp(E_\delta^d E_\delta) \left[ \int_0^\infty x^{s-1} e^{\frac{-1}{2}\beta x} W_{\mu,\nu}(\beta x) I_{p_i,q_i,r}^{m,n}[zx^k] dx \right] \frac{\prod_{j=1}^u \Gamma(A_j + \delta) C^\delta}{\prod_{j=1}^v \Gamma(B_j + \delta)}$$

$$= \exp(E_\delta^d E_\delta) \beta^{-s} I_{p_i+2,q_i+1,r}^{m,n+2} \left[ z\beta^{-k} \begin{vmatrix} (1/2 \pm \nu - s, k), (a_j, \alpha_j)_{1,n}, (a_{ji}, \alpha_{ji})_{n+1,p_i} \\ (b_j, \beta_j)_{1,m}, (b_{ji}, \beta_{ji})_{m+1,q_i}, (\mu - s, k) \end{vmatrix} \right]$$

$$\cdot \frac{\prod_{j=1}^u \Gamma(A_j + \delta) C^\delta}{\prod_{j=1}^v \Gamma(B_j + \delta)}. \quad (4.1)$$

Expanding both sides of (3.1) and using (2.2), we get

$$\sum_{h=0}^\infty \int_0^\infty x^{s-1} e^{\frac{-1}{2}\beta x} W_{\mu,\nu}(\beta x) I_{p_i,q_i,r}^{m,n}[zx^k] \frac{\prod_{j=1}^u \Gamma(A_j + \delta + h) C^{\delta+h}}{\prod_{j=1}^v \Gamma(B_j + \delta + h) h!} dx$$

$$= \beta^{-s} \frac{\prod_{j=1}^u \Gamma(A_j + \delta + h) C^{\delta+h}}{\prod_{j=1}^v \Gamma(B_j + \delta + h) h!}$$

$$\cdot I_{p_i+2,q_i+1,r}^{m,n+2} \left[ z\beta^{-k} \begin{vmatrix} (1/2 \pm \nu - dh - s, k), (a_j, \alpha_j)_{1,n}, (a_{ji}, \alpha_{ji})_{n+1,p_i} \\ (b_j, \beta_j)_{1,m}, (b_{ji}, \beta_{ji})_{m+1,q_i}, (\mu - dh - s, k) \end{vmatrix} \right]. \quad (4.2)$$

Now using (2.3) for changing the order of integration and summation on the left-hand side , which is justified due to [(2), p. 178 § 75 (III)] , and replacing  $A_j + \delta$  by  $A_j$  and  $B_j + \delta$  by  $B_j$  , we get (3.2).

#### 5. SPECIAL CASES:

1. On taking  $u = 2, v = 3, d = 1$  in (3.2) and using the result [5, pp.105,106]:

$${}_2F_1\left(\frac{1}{2}a + \frac{1}{2}b, \frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}; a, b, a + b - 1; 4x\right) \\ = {}_0F_1(-; a; x) {}_0F_1(-; b; x)$$

and

$${}_2F_3\left(a, b - a; b, \frac{1}{2}b, \frac{1}{2}b + \frac{1}{2}; \frac{1}{4}x^2\right) = {}_1F_1(a; b; x) {}_1F_1(a; b; -x)$$

we obtain

$$\int_0^\infty x^{s-1} e^{\frac{-1}{2}\beta x} W_{\mu, \nu}(\beta x) {}_0F_1\left(-; B_1; \frac{1}{4}Cx\right) {}_0F_1\left(-; B_2; \frac{1}{4}Cx\right) I_{p_i, q_i; r}^{m, n}[zx^k] dx \\ = \beta^{-s} \sum_{h=0}^\infty \frac{(B_1/2 + B_2/2)_h \left(\frac{B_1}{2} + \frac{B_2}{2} - \frac{1}{2}\right)_h C^h}{h! (B_1)_h (B_2)_h (B_1 + B_2 - 1)_h \beta^h} \\ \bullet I_{p_i+2, q_i+1, r}^{m, n+2} \left[ z\beta^{-k} \begin{vmatrix} (1/2 \pm \nu - dh - s, k), (a_j, \alpha_j)_{1, n}, (a_{ji}, \alpha_{ji})_{n+1, p_i} \\ (b_j, \beta_j)_{1, m}, (b_{ji}, \beta_{ji})_{m+1, q_i}, (\mu - dh - s, k) \end{vmatrix} \right] \quad (5.1)$$

and

$$\int_0^\infty x^{s-1} e^{\frac{-1}{2}\beta x} W_{\mu, \nu}(\beta x) {}_1F_1\left(A_1; B_1; 2(\sqrt{Cx})^2\right) \\ \times {}_1F_1(A_1; B_1 - 2; \sqrt{Cx}) I_{p_i, q_i; r}^{m, n}[zx^k] dx \\ = \beta^{-s} \sum_{h=0}^\infty \frac{(A)_h (B_1 - A_1)_h C^h}{h! (B_1)_h \left(\frac{1}{2}B_1\right)_h \left(\frac{1}{2}B_1 + \frac{1}{2}\right)_h \beta^h} \\ \bullet I_{p_i+2, q_i+1, r}^{m, n+2} \left[ z\beta^{-k} \begin{vmatrix} (1/2 \pm \nu - dh - s, k), (a_j, \alpha_j)_{1, n}, (a_{ji}, \alpha_{ji})_{n+1, p_i} \\ (b_j, \beta_j)_{1, m}, (b_{ji}, \beta_{ji})_{m+1, q_i}, (\mu - dh - s, k) \end{vmatrix} \right] \quad (5.2)$$

respectively. The conditions of validity for (5.1) and (5.2) are same as given with (3.1).

2. If we take  $\beta = d = C = u = v = 1, \mu = \nu + \frac{1}{2}$  in (3.2), replacing  $I$ -function on the right-hand side by its equivalent contour integral (1.1) then changing the order of summation and integration and evaluate the inner summation with the help of Gauss's theorem [1, p. 61], we find that

$$\begin{aligned}
 & \int_0^{\infty} x^{s-1} e^{-1/2x} W_{\nu+\frac{1}{2},\nu}(x) {}_1F_1(A_1; B_1; x) \\
 & \quad \times I_{p_i, q_i, r}^{m, n} \left[ z x^k \left| \begin{matrix} (a_j, \alpha_j)_{1, n}, (a_{ji}, \alpha_{ji})_{n+1, p_i} \\ (b_j, \beta_j)_{1, m}, (b_{ji}, \beta_{ji})_{m+1, q_i} \end{matrix} \right. \right] dx \\
 &= \frac{\Gamma(B_1)}{\Gamma(B_1 - A_1)} \\
 & \quad \times I_{p_i+2, q_i+1, r}^{m+1, n+1} \left[ z \left| \begin{matrix} (1/2-s-\nu, k), (a_j, \alpha_j)_{1, n}, (a_{ji}, \alpha_{ji})_{n+1, p_i}, (B_1-s-\nu-\frac{1}{2}, k) \\ (B_1-A_1-s-\nu-\frac{1}{2}, k), (b_j, \beta_j)_{1, m}, (b_{ji}, \beta_{ji})_{m+1, q_i} \end{matrix} \right. \right], \quad (5.3)
 \end{aligned}$$

provided

$$\begin{aligned}
 R\left(\rho + \frac{b_j}{\beta_j}\right) &> |R(\mu)| - \frac{1}{2}; \quad (j = 1, \dots, k), R\left(B_1 - A_1 - s - \nu - \frac{1}{2}\right) \\
 &> 0.
 \end{aligned}$$

3. Next taking  $\beta = C = d = u = v = 1, \alpha_1 = \frac{1}{2} + \lambda, \beta_1 = 1 + 2\lambda, \nu = 0$  in (3.2), replace  $s$  by  $s + \lambda - \frac{1}{2}$ ,  $I$ -function on the right-hand side by its equivalent contour integral (1.1), changing the order of summation and integration, evaluate the inner summation with the help of Watson theorem [1, p.189, eq. (6)] finally expressing  ${}_1F_1$  and Whittaker function  $W_{0,\nu}(x)$  in the modified Bessel function of the first and second kind respectively we obtain ,

$$\begin{aligned}
 & \int_0^{\infty} x^{s+\lambda-\frac{1}{2}-1} e^{-\frac{1}{2}x} W_{0,\nu}(x) {}_1F_1(A_1; B_1; x) \\
 & \quad \times I_{p_i, q_i, r}^{m, n} \left[ z x^k \left| \begin{matrix} (a_j, \alpha_j)_{1, n}, (a_{ji}, \alpha_{ji})_{n+1, p_i} \\ (b_j, \beta_j)_{1, m}, (b_{ji}, \beta_{ji})_{m+1, q_i} \end{matrix} \right. \right] dx \\
 &= \sqrt{\pi} \Gamma(\lambda + 1) \\
 & \quad \bullet I_{p_i+4, q_i+3, r}^{m+1, n+2} \left[ z \left| \begin{matrix} (1-\lambda-s \pm \nu, k), (a_j, \alpha_j)_{1, n}, (a_{ji}, \alpha_{ji})_{n+1, p_i}, \left(1 + \frac{\lambda}{2} \pm \frac{\nu}{2} - \frac{s}{2}, \frac{k}{2}\right) \\ (1-s, k), (b_j, \beta_j)_{1, m}, (b_{ji}, \beta_{ji})_{m+1, q_i}, \left(\frac{1}{2} - \frac{\lambda}{2} - \frac{s}{2} \pm \frac{\nu}{2}, \frac{k}{2}\right) \end{matrix} \right. \right], \quad (5.4)
 \end{aligned}$$

provided  $R\left(s + \lambda + \frac{b_j}{\beta_j}\right) > |R(\nu)|; \quad (j = 1, \dots, k), R(\lambda - s) > -\frac{1}{2}$ .

4. In (3.2), on taking  $r = 1$  the  $I$ -function reduces to Fox's  $H$ -function. Replacing  $k$  by  $m/t$  and by applying the following multiplication formula for  $H$ -function [6, p. 1029]:

$$H_{p,q}^{m,n} \left[ z x^{m/t} \left| \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right. \right] = (2\pi)^{(1-t)A} t^B H_{pt,qt}^{mt,nt} \left[ (zt^\tau)^t x^m \left| \begin{matrix} (\Delta(t, a_j), \alpha_j) \\ (\Delta(t, b_j), \beta_j) \end{matrix} \right. \right]$$

and by applying Gauss's multiplication theorem, we obtain a known result due to Singh [8]:

$$\begin{aligned} & \int_0^\infty x^{s-1} e^{-\frac{1}{2}\beta x} W_{\mu,\nu}(\beta x) {}_uF_v(A_u; B_u; Cx^d) H_{p,q}^{m,n} \left[ z x^{m/t} \left| \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right. \right] dx \\ &= (2\pi)^{(1-t)A + \frac{1}{2}(1-m)} t^B m^{s-\nu-\frac{1}{2}} \beta^{-s} \sum_{r=0}^\infty \frac{\prod_{j=1}^u (A_j)_r m^{rd} C^r}{\prod_{j=1}^v (B_j)_r \Gamma r \beta^{rd}} \\ & \cdot H_{pl+2m, ql+m}^{ml, nl+2m} \left[ \frac{(zl^\tau)^t m^m}{\beta^m} \left| \begin{matrix} (\Delta(m, \frac{1}{2} - s - rd \pm \nu), 1), \{(\Delta(t, a_j), \alpha_j)\} \\ \{(\Delta(t, b_j), \beta_j)\}, (\Delta(m, \mu - s - rd, 1)) \end{matrix} \right. \right], \quad (5.5) \end{aligned}$$

where

$$\begin{aligned} A &= k + l - \frac{p}{2} - \frac{q}{2}, B = \sum_{j=1}^q b_j - \sum_{j=1}^p a_j + \frac{p}{2} - \frac{q}{2} + 1, \tau \\ &= \sum_{j=1}^p e_j - \sum_{j=1}^q f_j, \end{aligned}$$

$\{(\Delta(t, \delta_r), \gamma_r)\}$  stands for  $\left\{\left(\frac{\delta_r}{t}, \gamma_r\right)\right\}, \dots, \dots, \left\{\left(\frac{\delta_{r+t-1}}{t}, \gamma_r\right)\right\}$ ;  $m, d$  and  $t$  positive integers and  $(\Delta(t, \pm\alpha), 1)$  denotes  $\{(\Delta(t, \alpha), 1)\}, (\Delta(t, -\alpha), 1)$ .

The integrals (5.1), (5.2) and (5.3) give some very interesting cases by reducing some or all the functions occurring in the integrand on the left. By proper choice of parameters  ${}_0F_1$  can be reduced to a Bessel function and can also be transformed to  ${}_1F_1$  by using Kummer's second theorem [5, p. 126] the  ${}_1F_1$  can be reduced to a Whittaker function  $M_{k,m}(x)$ , generalized Laguerre polynomials  $L_n^\alpha(x)$ , Weber's parabolic cylinder function  $D_n(x)$  and modified Bessel function of the first kind  $I_n(x)$ .

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## APPLICATION OF FOX'S H-FUNCTION IN ELECTRIC CIRCUIT THEORY

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### ABSTRACT

The aim of this paper is to obtain the value of the charge at any time  $t$  in a simple electric circuit consisting of resistance, inductance, capacitance and a source of electromotive force  $E_0P(t)$ , when  $P(t)$  is taken in terms of the Fox's H-functions. This function is believed to be quite general nature because it includes a number of well known elementary functions as its particular cases. Evidently, therefore, our results would apply to a wide variety of useful functions (or products of several such functions) occurring frequently in mathematical physics and engineering.

### 1. INTRODUCTION:

The H-function of one variable [8, p.10] is defined as:

$$H_{p,q}^{m,n} [x | \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix}] = (1/2\pi i) \int_L \theta(s) x^s ds$$

where  $i = \sqrt{-1}$ ,

$$\theta(s) = \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j s) \prod_{j=1}^n \Gamma(1 - a_j + \alpha_j s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + \beta_j s) \prod_{j=n+1}^p \Gamma(a_j - \alpha_j s)}$$

where

$$\sum_{j=1}^n \alpha_j - \sum_{j=n+1}^p \alpha_j + \sum_{j=1}^m \beta_j - \sum_{j=m+1}^q \beta_j \equiv M > 0,$$

and  $|\arg x| < \frac{1}{2} M\pi$ .

The following integral which is a special case of [2] will be required in the sequel

$$\begin{aligned} & \int_0^t x^{\rho-1} (c + bx)^{-\lambda} e^{Rx/2L} \sin\{\omega(t-x)\} H_{P,Q}^{M,N} [zx^u (c + bx)^\epsilon]_{(b_j, B_j)_{1,Q}}^{(a_j, A_j)_{1,P}} dx \\ &= t^{\rho+1} c^{-\lambda} \sum_{r,k,m=0}^{\infty} \frac{(-1)^k (\omega)^{2k+1} (t)^{2k+m} (-b/c)^m (Rt/2L)^r}{r!m!} \\ & \times H_{P+2,Q+2}^{M+1,N+1} [zt^u c^\epsilon]_{(\lambda+m,\epsilon),(b_j, B_j)_{1,Q},(-1-2k-\rho-r-m,u)}^{(1-\rho-r-m,u),(a_j, A_j)_{1,P},(\lambda,\epsilon)} \end{aligned} \quad (1.1)$$

$$\begin{aligned} &= t^{\rho+1} c^{-\lambda} \sum_{r,k=0}^{\infty} \sum_{m=0}^r \frac{(-1)^k (\omega)^{2k+1} (t)^{2k+r} (b/c)^m (R/2L)^{r-m} (-r)_m}{r!m!} \\ & \times H_{P+2,Q+2}^{M+1,N+1} [zt^u c^\epsilon]_{(\lambda+m,\epsilon),(b_j, B_j)_{1,Q},(-1-2k-\rho-r,u)}^{(1-\rho-r,u),(a_j, A_j)_{1,P},(\lambda,\epsilon)} \end{aligned} \quad (1.2)$$

$$\begin{aligned} &= t^{\rho+1} c^{-\lambda} e^{Rt/2L} \sum_{r=0}^{\infty} \sum_{k=0}^{[r/2]} \sum_{m=0}^{r-2k} \frac{(-1)^k (\omega)^{2k-1} (t)^r (-b/c)^m \Gamma(2+r-m)}{\Gamma(2+2k)(r-2k-m)!m!} \\ & \times H_{P+2,Q+2}^{M+1,N+1} [zt^u c^\epsilon]_{(\lambda+m,\epsilon),(b_j, B_j)_{1,Q},(-1-\rho-r,u)}^{(1-\rho-m,u),(a_j, A_j)_{1,P},(\lambda,\epsilon)} (-R/2L)^{r-2k-m} \end{aligned} \quad (1.3)$$

Throughout this paper  $H_{P,Q}^{M,N} [z]_{(b_j, B_j)_{1,Q}}^{(a_j, A_j)_{1,P}}$  stands for the well-known Fox's H-function. An interesting and useful account of this function can be found, for example, in [4] and [7]. Here  $(a_j, A_j)_{1,P}$  abbreviates the parameter sequence  $(a_1, A_1), \dots, (a_P, A_P)$ , and so on. Also, the symbol  $[r/2]$  stands for the greatest integer in  $r/2$ .

The conditions of validity of the integral are  $\text{Re}(\lambda) > 0$ ,  $\min(u, \epsilon) > 0$ ,  $t > 0$ ,  $|bt/c| < 1$ ,  $A > 0$ ,  $|\arg z| < (1/2) A\pi$ ,  $\text{Re}(\rho) + u \min_{1 \leq j \leq M} \{\text{Re}(b_j/B_j)\} > 0$  and the series on the right-hand side converges absolutely, it being understood that

$$A = \sum_{j=1}^N A_j - \sum_{j=N+1}^P A_j + \sum_{j=1}^M B_j - \sum_{j=M+1}^Q B_j \quad (1.4)$$

### Proof of (1.2):

To prove (1.2), we make use of known results [4, p.2, (1.1.1)] and [5, p.58, (3)] in (1.1).

**Proof of (1.3):**

On using the contour representation for Fox's H-function occurring on the right-hand side of (1.1), changing the order of integration and summation with respect to  $r$ , we find that R. H. S. of (1.1).

$$\begin{aligned}
 &= t^{\rho+1} c^{-\lambda} \sum_{k,m=0}^{\infty} \frac{(-1)^k (\omega)^{2k+1} (t)^{2k+m} (-b/c)^m}{m!} \\
 &\times \left(\frac{1}{2\pi i}\right) \int_L \prod_{j=1}^M \Gamma(b_j - B_j s) \prod_{j=1}^N \Gamma(1 - a_j + A_j s) \\
 &\times [\prod_{j=M+1}^Q \Gamma(1 - b_j + B_j s) \prod_{j=N+1}^P \Gamma(a_j - A_j s)]^{-1} \\
 &\times (zt^u c^\epsilon)^s \frac{\Gamma(\lambda+m-\epsilon s) \Gamma(\rho+m+us)}{\Gamma(\lambda-\epsilon s) \Gamma(2+2k+\rho+m+us)} \\
 &\times {}_1F_1[\rho+m+us; 2+2k+\rho+m+us; Rt/2L] ds \tag{1.5}
 \end{aligned}$$

Now using Kummer's first formula [5, p.125, eqn. (2)] in (1.5) and then applying series representation for  ${}_1F_1$  thus obtained, we easily get

$$\begin{aligned}
 \text{R.H.S. of (1.1)} &= t^{\rho+1} c^{-\lambda} e^{Rt/2L} \sum_{r,k,m=0}^{\infty} \frac{(-1)^k (\omega)^{2k+1} (t)^{2k+m}}{r!m!} \\
 &\times (-b/c)^m (Rt/2L)^r (2+2k)_r \\
 &\times H_{P+2,Q+2}^{M+1,N+1} [zt^u c^\epsilon]_{(\lambda+m,\epsilon),(b_j,B_j)_{1,Q},(-1-2k-\rho-r-m,u)}^{(1-\rho-m,u),(a_j,A_j)_{1,P},(\lambda,\epsilon)} \tag{1.6}
 \end{aligned}$$

Finally, applying to the results [5, p. 56, eq. (1); p.57, eqn. (7)] in (1.6), we arrive at (1.3).

**2. MAIN PROBLEM:**

If we consider an electric circuit consisting of resistance  $R$ , an inductance  $L$ , a condenser of capacity  $C$  and a source of electromotive force  $E_0 p(t)$ , where  $E_0$  is constant and  $P(t)$  is known function of time  $t$ , the charge  $q(t)$  on the plates of condenser at any time  $t$ , satisfies the following second order differential equation

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{c} = E_0 P(t) \tag{2.1}$$

The solution of this differential equation subject to initial conditions  $q = Q$ ,  $i = dq/dt = I$  when  $t = 0$ , is the standard result [6, p.95] and is given below

$$q(t) = J(t) + \frac{E_0}{\omega L} e^{-Rt/2L} \int_0^t P(\eta) e^{R\eta/2L} \sin\{\omega(t - \eta)\} d\eta \quad (2.2)$$

where, for convenience,

$$J(t) = e^{-Rt/2L} [Q \cos \omega t + (I_1/\omega) \sin \omega t] \quad (2.3)$$

$$I_1 = I + RQ/2L \text{ and } (1/LC) - (R^2/4L^2) = \omega^2 > 0. \quad (2.4)$$

Now we turn to the problem of finding out the charge  $q(t)$  when  $P(t)$  is taken in terms of the H-function. We shall discuss two useful cases.

(i) *Solution of (2.1) when  $P(t)$  is taken in terms of the H-function*

Let

$$P(t) = t^{\rho-1} (c + bt)^{-\lambda} H_{P,Q}^{M,N} [zt^u (c + bt)^{\epsilon}]_{(b_j, B_j)_{1,Q}}^{(a_j, A_j)_{1,P}} \quad (2.5)$$

Putting the above value of  $P(t)$  in (2.2) and evaluating the  $\square$ -integral with the help of (1.1) {or (1.2) or (1.3)}, we find that the value of the charge  $q(t)$  is given by

$$q(t) = J(t) + \frac{E_0}{\omega L} e^{-\frac{Rt}{2L}} F_i(r, k, m, t), i = 1, 2, 3 \quad (2.6)$$

where  $j(t)$ ,  $F_i(r, k, m, t)$ , ( $i = 1, 2, 3$ ) stand for the quantities as given by (2.3), (1.1), (1.2) and (1.3) respectively and the conditions mentioned after (1.3) are satisfied.

It is interesting to note that the value of the current  $dq/dt$  can also be obtained from (2.6), by differentiating the series on its right-hand side term by term with respect to  $t$ . The process of term by term differentiation is assumed to be justified as the H-function being analytic function [4, p.3] and the resulting series of H-functions obtained in this case will be uniform convergent in any arbitrary domain  $0 \leq t \leq a$ .

A special case of the solution (2.6), which is of practical interest, follows easily by putting  $R = 0$ ; thus we arrive at the following solution

$$\begin{aligned} q(t) = & Q \cos \omega t + (I/\omega) \sin \omega t \\ & + c^{-\lambda} \sum_{k,m=0}^{\infty} \frac{(-1)^k (\omega)^{2k} (t)^{2k+m} (-b/c)^m E_0 t^{\rho+1}}{m! L} \\ & \times H_{P+2, Q+2}^{M+1, N+1} [zt^u c^{\epsilon}]_{(\lambda+m, \epsilon), (b_j, B_j)_{1, Q}, (-1-2k-\rho-m, u)}^{(1-\rho-m, u), (a_j, A_j)_{1, P}, (\lambda, \epsilon)} \end{aligned} \quad (2.7)$$

(ii) Solution of (2.1), when  $P(t)$  is a series of Sines:

If we substitute  $\eta = \sin\theta$  in (2.2), the values of the charge  $q(t)$  is easily seen to be given by the following equivalent form

$$q(t) = J(t) + \frac{E_0}{\omega L} e^{-Rt/2L} \int_0^{\sin^{-1}t} P(\sin\theta) e^{R\sin\theta/2L} \sin\{\omega(t - \sin\theta)\} \cos\theta d\theta \quad (2.8)$$

Now on taking

$$P(\sin t) = \sum_{r=0}^{\infty} H_{P+2,Q+2}^{M+1,N+1} [z]_{(3/2-\rho,h),(b_j,B_j)_{1,Q}(1-\rho,h)}^{(1-\rho-r,h),(a_j,A_j)_{1,P}(2-\rho+r,h)}]$$

$$\sin(2r+1)t = (\sqrt{\pi}/2) (\sin t)^{1-2\rho} H_{P,Q}^{M,N} [z(\sin t)^{-2h}]_{(b_j,B_j)_{1,Q}}^{(a_j,A_j)_{1,P}}] \quad (2.9)$$

[by virtue of a known result [1, p.705]

in (2.8) and again replacing  $\sin \theta$  by  $x$ , we find that

$$q(t) = J(t) + \frac{E_0\sqrt{\pi}}{2\omega L} e^{-Rt/2L} \int_0^t x^{1-2\rho} e^{Rx/2L} \sin\{\omega(t-x)\} H[zx^{-2h}] dx \quad (2.10)$$

Now appealing to a property of the H-function [4, p.4 (1.2.2)] and evaluating the integral with the help of (1.1) [or (1.3)] with  $c = 1$ ,  $b, \epsilon \rightarrow 0$ , we get the following value of  $q(t)$ :

$$q(t) = J(t) + \frac{E_0(t)^{3-2\rho}\sqrt{\pi}}{2L} e^{-Rt/2L} \sum_{r,k=0}^{\infty} \frac{(-\omega^2 t^2)^k (Rt/2L)^r}{r!}$$

$$\times H_{P+1,Q+1}^{M+1,N} [z(t)^{-2h}]_{(2-2\rho+r,2h),(b_j,B_j)_{1,Q}}^{(a_j,A_j)_{1,P}(4-2\rho+r+2k,2h)}] \quad (2.11)$$

$$= J(t) + \frac{E_0(t)^{3-2\rho}\sqrt{\pi}}{2L} \sum_{r=0}^{\infty} (r+1) B_r (-Rt/2L)^r$$

$$\times H_{P+1,Q+1}^{M+1,N} [z(t)^{-2h}]_{(2-2\rho+r,2h),(b_j,B_j)_{1,Q}}^{(a_j,A_j)_{1,P}(4-2\rho+r,2h)}] \quad (2.12)$$

where  $J(t)$  is given by (2.3) and

$$B_r = \sum_{k=0}^{[r/2]} \frac{(-1)^k (-r)_{2k} (2L\omega/R)^{2k}}{(2)_{2k}} \quad (2.13)$$

The conditions of validity of (2.11) and (2.12) are

(i)  $h > 0, t > 0, A > 0, |\arg z| < (1/2)A\pi$  {A is given by (1.4)}.

(ii)  $\operatorname{Re}(1 - \rho) + h \min_{1 \leq j \leq N} \{\operatorname{Re}(1 - a_j)/A_j\} > 0$ .

Thus, when the electromotive force  $E_0 P(t)$  is in the form of series of sines:

$$E_0 \sum_{r=0}^{\infty} H_{P+2, Q+2}^{M+1, N+1} [z]_{(3/2-\rho, h), (b_j, B_j)_{1, Q}, (1-\rho, h)}^{(1-\rho-r, h), (a_j, A_j)_{1, P}, (2-\rho+r, h)} \sin (2r + 1) t \quad (2.14)$$

the value of  $q(t)$  is given by (2.11) {or (2.12)}.

The solution of (2.1) given by [3, p.738, (3.4)] is contained in our solution (2.11). This can be verified easily by putting  $R = 0, \rho = 0, h = 1, A_i = B_j = 1$  ( $i = 1, \dots, P; j = 1, \dots, Q$ ) in (2.11) and appealing to the Gauss multiplication formula therein.

### 3. PARTICULAR CASE:

The solutions of (2.1) are quite general in character as these possess twofold generality. The one is the general nature of the H-function and second is exhibited by the presence of the general arguments in this function. By making a free use of results [4, p. 145 to 151], our solutions can be suitably applied to a remarkable wide variety of useful functions (or product of such functions) that occur frequently in the problems of mathematical physics and engineering. Here we mention only some interesting special cases of the solution of (2.1) given by (2.6).

On taking  $c = 1$  and  $b \rightarrow 0$  in (2.5) and (2.6), we find that

$$P(t) = (t)^{\rho-1} H_{P, Q}^{M, N} [z(t)^u]_{(b_j, B_j)_{1, Q}}^{(a_j, A_j)_{1, P}} \quad (3.1)$$

then

$$q(t) = J(t) + \frac{E_0(t)^{\rho+1}}{L} e^{-Rt/2L} \sum_{r, k=0}^{\infty} \frac{(-\omega^2 t^2)^k (Rt/2L)^r}{r!} \\ \times H_{P+1, Q+1}^{M, N+1} [z(t)^{-2h}]_{(b_j, B_j)_{1, Q}, (-1-2k-\rho-r, u)}^{(1-\rho-r, u), (a_j, A_j)_{1, P}} \quad (3.2)$$

or equivalently,

$$q(t) = J(t) + \frac{E_0(t)^{\rho+1}}{L} \sum_{r=0}^{\infty} (r+1) B_r (-Rt/2L)^r \\ \times H_{P+1, Q+1}^{M, N+1} [zt^u]_{(b_j, B_j)_{1, Q}, (-1-\rho-r, u)}^{(1-\rho, u), (a_j, A_j)_{1, P}} \quad (3.2)$$

where  $J(t)$  and  $B_r$  are given by (2.3) and (2.13) respectively. The conditions of validity for the solution can be easily obtained from those given after (1.3).

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## APPLICATION OF MULTIVARIABLE H-FUNCTION IN THE FIELD OF PHOTOSYNTHESIS

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### ABSTRACT

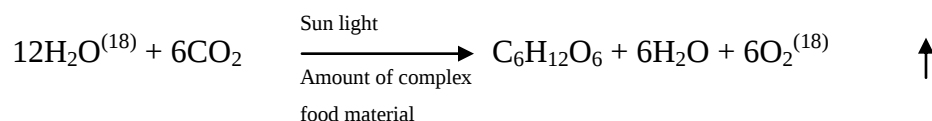
In this Paper, we represent that process of Photosynthesis in terms of multivariable H-function, where the result has been performed with respect to their parameters (these are mainly H<sub>2</sub>O, CO<sub>2</sub>, sunlight and chlorophyll) of the multivariable H-function. On specializing the parameters of the multivariable H-function involved in the main result, many new and unknown results may be obtained.

### 1. INTRODUCTION:

Photosynthesis is the single most important physico-biochemical process of the world on which the existence of life on earth depends. It is the ability of green plants only that they utilize the energy of the sun light to produce oxygen containing organic material from stable inorganic matter by Photosynthesis process.

Generally, the Photosynthesis is the Physico-Biochemical process, which produces complex carbohydrates by reaction of water and carbon-dioxide in the presence of light and chlorophyll.

According to Ruban, Randel and Kamel during the Photosynthesis oxygen and heavy isotopes of water O<sup>(18)</sup> and H<sub>2</sub>O<sup>(18)</sup> react and produce O<sub>2</sub>, which is obtained by H<sub>2</sub>O. This reaction produces complex hydrocarbon's with water Tyagi [5].



Four parameters works to complete this reaction out of which two supports internally and two externally. In this way the change occurs in parameters with respect to temperature (6<sup>0</sup>C to 37<sup>0</sup>C) these are represented symbolically in the following way and the internal change of variables depends upon these abbreviations given by Lax [3].

$$\begin{array}{c}
 \begin{array}{ccc}
 \downarrow & & \uparrow \\
 \alpha + \gamma & \xrightarrow{\text{External}} & \frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)} + 6\text{H}_2\text{O} + 6\text{O}_2 \\
 \uparrow & & \downarrow \\
 \beta & & \delta \\
 \text{Internal} & & \\
 \text{Parameter} & & 
 \end{array}
 \end{array}
 \quad \begin{array}{c}
 \boxed{\text{Amount of complex food material}}
 \end{array}
 \quad (1)$$

Here water is evaporated and  $\text{O}_2$  goes to atmosphere we compare it with the formula given by Erdelyi [2] for simplification of temperature. See Bhatnagar [1, p.11-16] and Lax [3].

$$\begin{aligned}
 & \frac{1}{2\pi i} \int_{t-i\infty}^{t+i\infty} [\Gamma(\alpha+t)\Gamma(\beta+t)\Gamma(\gamma-t)\Gamma(\delta-t)] dt \\
 &= \frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)} + A, \quad (2)
 \end{aligned}$$

where  $\text{Re}(\alpha) > 0$ ,  $\text{Re}(\beta) > 0$ ,  $\text{Re}(\gamma) > 0$ ,  $\text{Re}(\delta) > 0$ , and  $\alpha, \beta, \gamma, \delta$  are respectively denote the parameters water ( $\alpha$ ), carbon-dioxide ( $\beta$ ), sunlight ( $\gamma$ ) and chlorophyll ( $\delta$ ).  $A = \text{const.}$  when  $\alpha = \beta = \gamma = \delta = 0$ , then  $A$  will be 0.

The multivariable H-function given in [4] is defined as follows:

$$\begin{aligned}
 H[z_1, \dots, z_r] &= H_{p,q;p_1,q_1;\dots;p_r,q_r}^{0,n;m_1,n_1;\dots;m_r,n_r} \left[ \begin{array}{c} z_1 \\ \vdots \\ z_r \end{array} \middle| \begin{array}{c} (a_j; \alpha_j', \dots, \alpha_j^{(r)})_{1,p} : (c_j'; \gamma_j')_{1,p_1}; \dots; (c_j^{(r)}; \gamma_j^{(r)})_{1,p_r} \\ (b_j; \beta_j', \dots, \beta_j^{(r)})_{1,q} : (d_j'; \delta_j')_{1,q_1}; \dots; (d_j^{(r)}; \delta_j^{(r)})_{1,q_r} \end{array} \right] \\
 &= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \phi_1(\xi_1) \dots \phi_r(\xi_r) \psi(\xi_1, \dots, \xi_r) z_1^{\xi_1} \dots z_r^{\xi_r} d\xi_1 \dots d\xi_r \quad (3)
 \end{aligned}$$

where  $\omega = \sqrt[3]{(-1)}$ ,

$$\begin{aligned}
 \psi(\xi_1, \dots, \xi_r) &= \frac{\prod_{j=1}^n \Gamma(1-a_j + \sum_{i=1}^r \alpha_j^{(i)} \xi_i)}{\prod_{j=n+1}^p \Gamma(a_j - \sum_{i=1}^r \alpha_j^{(i)} \xi_i) \prod_{j=1}^q \Gamma(1-b_j + \sum_{i=1}^r \beta_j^{(i)} \xi_i)} \\
 \phi_i(\xi_i) &= \frac{\prod_{j=1}^{m_i} \Gamma(d_j^{(i)} - \delta_j^{(i)} \xi_i) \prod_{j=1}^{n_i} \Gamma(1-c_j^{(i)} + \gamma_j^{(i)} \xi_i)}{\prod_{j=m_i+1}^{q_i} \Gamma(1-d_j^{(i)} + \delta_j^{(i)} \xi_i) \prod_{j=n_i+1}^{p_i} \Gamma(c_j^{(i)} - \gamma_j^{(i)} \xi_i)}
 \end{aligned}$$

In (3),  $i$  in the superscript  $(i)$  stands for the number of primes, e.g.,  $b^{(1)} = b'$ ,  $b^{(2)} = b''$ , and so on; and an empty product is interpreted as unity.

Suppose, as usual, that the parameters

$$\begin{aligned}
 & a_j, j = 1, \dots, p; c_j^{(i)}, j = 1, \dots, p_i; \\
 & b_j, j = 1, \dots, q; d_j^{(i)}, j = 1, \dots, q_i; \forall i \in \{1, \dots, r\}
 \end{aligned}$$

are complex numbers and the associated coefficients

$$\alpha_j^{(i)}, j = 1, \dots, p; \gamma_j^{(i)}, j = 1, \dots, p_i;$$

$$\beta_j^{(i)}, j = 1, \dots, q; \delta_j^{(i)}, j = 1, \dots, q_i; \forall i \in \{1, \dots, r\}$$

positive real numbers such that the left of the contour. Also

$$V_i = \sum_{j=1}^p \alpha_j^{(i)} + \sum_{j=1}^{p_i} \gamma_j^{(i)} - \sum_{j=1}^q \beta_j^{(i)} - \sum_{j=1}^{q_i} \delta_j^{(i)} \leq 0 \quad (4)$$

$$\Omega_i = - \sum_{j=n+1}^p \alpha_j^{(i)} - \sum_{j=1}^q \beta_j^{(i)} + \sum_{j=1}^{m_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} \delta_j^{(i)} + \sum_{j=1}^{n_i} \gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \gamma_j^{(i)} > 0 \quad (5)$$

where the integral  $n$ ,  $p$ ,  $q$ ,  $m_i$ ,  $n_i$ ,  $p_i$  and  $q_i$  are constrained by the inequalities  $p \geq n \geq 0$ ,  $q \geq 0$ ,  $q_i \geq m_i \geq 1$  and  $p_i \geq n_i \geq 1 \forall i \in \{1, 2, \dots, r\}$  and the inequalities in (4) hold for suitably restricted values of the complex variables  $z_1, \dots, z_r$ . The sequence of parameters in (1) are such that none of the poles of the integrand coincide, that is, the poles of the integrand in (1) are simple. The contour  $L_i$  in the complex  $\xi_i$ -plane is of the Mellin-Barnes type which runs from  $-\omega\infty$  to  $+\omega\infty$  with indentations, if necessary, to ensure that all the poles of  $\Gamma(d_j^{(i)} - \delta_j^{(i)}\xi_i)$ ,  $j = 1, \dots, m_i$  are separated from those of  $\Gamma(1 - c_j^{(i)} + \gamma_j^{(i)}\xi_i)$ ,  $i = 1, \dots, n_i$ .

## 2. MAIN RESULT:

In this section the parameters (water, carbon-dioxide, sunlight and chlorophyll) are complete individually. At that condition the Photosynthesis is represented in multivariable H-function as follows:

$$\frac{1}{2\pi i} \int_{t-i\infty}^{t+i\infty} H_{p, q}^{0, n} \left[ \begin{matrix} (m_1, n_1+4); \dots; (m_r, n_r) \\ (p_1+4, q_1); \dots; (p_r, q_r) \end{matrix} \right] \left[ \begin{matrix} Z_1 \\ Z_r \end{matrix} \right] \dots \left[ \begin{matrix} (1-\alpha-t, \alpha_1), (1-\beta-t, \beta_1), (1-\gamma-t, \gamma_1), (1-\delta-t, \delta_1), \dots \\ \dots \end{matrix} \right] dt$$

$$= H_{p, q}^{0, n} \left[ \begin{matrix} (m_1, n_1+4); \dots; (m_r, n_r) \\ (p_1+4, q_1+1); \dots; (p_r, q_r) \end{matrix} \right] \left[ \begin{matrix} Z_1 \\ Z_r \end{matrix} \right] \dots \left[ \begin{matrix} (1-\alpha-\gamma, \alpha_1+\gamma_1), (1-\alpha-\delta, \alpha_1+\delta_1), (1-\beta-\gamma, \beta_1+\gamma_1), (1-\beta-\delta, \beta_1+\delta_1), \dots \\ \dots, (1-\alpha-\beta-\gamma-\delta, \alpha_1+\beta_1+\gamma_1+\delta_1); \dots \end{matrix} \right] + A, \quad (6)$$

$|\arg(z_k)| < \frac{1}{2} V_k \pi, \forall k \in [1, \dots, r]$ , where  $V_k$  is given in (4).

## 3. ANALYSIS AND PROOF OF MATHEMATICAL FORMULA:

To prove (6), we put the value of internal and external parameters for Photosynthesis and for a healthy plant whose surrounding temperature is  $6^\circ$  to  $36^\circ$ ,  $\alpha = \alpha + \alpha_1 \xi_1, \beta = \beta + \beta_1 \xi_1, \gamma = \gamma + \gamma_1 \xi_1, \delta = \delta + \delta_1 \xi_1$  (see Tyagi [5] and Bhatnagar [1]) in the integral of (2) on the both sides and multiplying by  $\frac{1}{(2\pi\omega)^r} [\phi_1(\xi_1) \dots \phi_r(\xi_r) \psi(\xi_1, \dots, \xi_r)] z_1^{\xi_1} \dots z_r^{\xi_r}$  both sides, and further integrating in the direction of  $L_1, \dots, L_r$  with respect to time and after changing the order of integration on the left hand side, we get the required result (6).

#### 4. APPLICATION:

In this section, we remove various parameters (water, carbon-dioxide, sunlight and chlorophyll) in the reaction of Photosynthesis respectively and represent their position by formula.

##### 1. Photosynthesis in the absence of water:

If we put  $\alpha_1 = 0$  in the main result then we get following formula:

$$\begin{aligned} & \frac{1}{2\pi i} \int_{t-i\infty}^{t+i\infty} H_{p, q}^{0, n} \left[ \begin{matrix} (m_1, n_1+3); \dots; (m_r, n_r) \\ (p_1+3, q_1); \dots; (p_r, q_r) \end{matrix} \middle| \begin{matrix} Z_1 \\ Z_r \end{matrix} \right] \Gamma(\alpha + t) dt \\ &= H_{p, q}^{0, n} \left[ \begin{matrix} (m_1, n_1+4); \dots; (m_r, n_r) \\ (p_1+4, q_1+1); \dots; (p_r, q_r) \end{matrix} \middle| \begin{matrix} Z_1 \\ Z_r \end{matrix} \right] \Gamma(\alpha + t) dt \\ &+ A, \\ &|\arg(z_k)| < \frac{1}{2} V_k \pi, \forall k \in [1, \dots, r], \text{ where } V_k \text{ is given in (4)}. \end{aligned} \quad (7)$$

##### 2. Photosynthesis in the absence of carbon-dioxide:

If we put  $\beta_1 = 0$  in the main result then we get following formula:

$$\begin{aligned} & \frac{1}{2\pi i} \int_{t-i\infty}^{t+i\infty} H_{p, q}^{0, n} \left[ \begin{matrix} (m_1, n_1+3); \dots; (m_r, n_r) \\ (p_1+3, q_1); \dots; (p_r, q_r) \end{matrix} \middle| \begin{matrix} Z_1 \\ Z_r \end{matrix} \right] \Gamma(\beta + t) dt \\ &= H_{p, q}^{0, n} \left[ \begin{matrix} (m_1, n_1+4); \dots; (m_r, n_r) \\ (p_1+4, q_1+1); \dots; (p_r, q_r) \end{matrix} \middle| \begin{matrix} Z_1 \\ Z_r \end{matrix} \right] \Gamma(\beta + t) dt \\ &+ A, \\ &|\arg(z_k)| < \frac{1}{2} V_k \pi, \forall k \in [1, \dots, r], \text{ where } V_k \text{ is given in (4)}. \end{aligned} \quad (8)$$

##### 3. Photosynthesis in the absence of sunlight:

If we put  $\gamma_1 = 0$  in the main result then we get following formula:

$$\begin{aligned}
 & \frac{1}{2\pi i} \int_{t-i\infty}^{t+i\infty} H_{p, q}^{0, n} \left[ \begin{matrix} (m_1, n_1+3); \dots; (m_r, n_r) \\ (p_1+3, q_1); \dots; (p_r, q_r) \end{matrix} \right] \begin{matrix} Z_1 \\ Z_r \end{matrix} \\
 & \quad \dots, \dots, (1-\alpha-t, \alpha_1), (1-\beta-t, \beta_1), (1-\delta-t, \delta_1), \dots, \dots \Gamma(\gamma+t) dt \\
 & = H_{p, q}^{0, n} \left[ \begin{matrix} (m_1, n_1+4); \dots; (m_r, n_r) \\ (p_1+4, q_1+1); \dots; (p_r, q_r) \end{matrix} \right] \begin{matrix} Z_1 \\ Z_r \end{matrix} \\
 & \quad \dots, \dots, (1-\alpha-\gamma, \alpha_1), (1-\alpha-\delta, \alpha_1+\delta_1), (1-\beta-\gamma, \beta_1), (1-\beta-\delta, \beta_1+\delta_1), \dots, \dots \\
 & \quad \dots, \dots, (1-\alpha-\beta-\gamma-\delta, \alpha_1+\beta_1+\delta_1); \dots, \dots \Gamma(\gamma+t) dt \quad (9) \\
 & + A, \\
 & |\arg(z_k)| < \frac{1}{2} V_k \pi, \forall k \in [1, \dots, r], \text{ where } V_k \text{ is given in (4).}
 \end{aligned}$$

#### 4. Photosynthesis in the absence of chlorophyll:

If we put  $\delta_1 = 0$  in the main result then we get following formula:

$$\begin{aligned}
 & \frac{1}{2\pi i} \int_{t-i\infty}^{t+i\infty} H_{p, q}^{0, n} \left[ \begin{matrix} (m_1, n_1+3); \dots; (m_r, n_r) \\ (p_1+3, q_1); \dots; (p_r, q_r) \end{matrix} \right] \begin{matrix} Z_1 \\ Z_r \end{matrix} \\
 & \quad \dots, \dots, (1-\alpha-t, \alpha_1), (1-\beta-t, \beta_1), (1-\gamma-t, \gamma_1), \dots, \dots \Gamma(\delta+t) dt \\
 & = H_{p, q}^{0, n} \left[ \begin{matrix} (m_1, n_1+4); \dots; (m_r, n_r) \\ (p_1+4, q_1+1); \dots; (p_r, q_r) \end{matrix} \right] \begin{matrix} Z_1 \\ Z_r \end{matrix} \\
 & \quad \dots, \dots, (1-\alpha-\gamma, \alpha_1+\gamma_1), (1-\alpha-\delta, \alpha_1), (1-\beta-\gamma, \beta_1+\gamma_1), (1-\beta-\delta, \beta_1), \dots, \dots \\
 & \quad \dots, \dots, (1-\alpha-\beta-\gamma-\delta, \alpha_1+\beta_1+\gamma_1); \dots, \dots \Gamma(\delta+t) dt \quad (10) \\
 & + A, \\
 & |\arg(z_k)| < \frac{1}{2} V_k \pi, \forall k \in [1, \dots, r], \text{ where } V_k \text{ is given in (4).}
 \end{aligned}$$

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## HEAT CONDUCTION IN A SQUARE PLATE INVOLVING GENERALIZED H-FUNCTION OF TWO VARIABLES

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### ABSTRACT

The aim of this paper is to obtain a solution of a simple problem of heat conduction in a square plate with the help of generalized H-function of two variables.

### 1. INTRODUCTION:

The generalized H-function of two variables is given by Shrivastava, H. S. P. [3] and defined as follows:

$$H_{p_1, q_1; p_2, q_2; p_3, q_3}^{m_1, n_1; m_2, n_2; m_3, n_3} \left[ \begin{matrix} x \\ y \end{matrix} \middle| \begin{matrix} (a_j, \alpha_j, A_j)_{1, p_1}; (c_j, \gamma_j)_{1, p_2}; (e_j, E_j)_{1, p_3} \\ (b_j, \beta_j, B_j)_{1, q_1}; (d_j, \delta_j)_{1, q_2}; (f_j, F_j)_{1, q_3} \end{matrix} \right]$$

$$= \frac{-1}{4\pi^2} \int_{L_1} \int_{L_2} \phi_1(\xi, \eta) \theta_2(\xi) \theta_3(\eta) x^\xi y^\eta d\xi d\eta, \quad (1)$$

where

$$\phi_1(\xi, \eta) = \frac{\prod_{j=1}^{n_1} \Gamma(1 - a_j + \alpha_j \xi + A_j \eta) \prod_{j=1}^{m_1} \Gamma(b_j - \beta_j \xi - B_j \eta)}{\prod_{j=1}^{p_1} \Gamma(a_j - \alpha_j \xi - A_j \eta) \prod_{j=1}^{q_1} \Gamma(1 - b_j + \beta_j \xi + B_j \eta)},$$

$$\theta_2(\xi) = \frac{\prod_{j=1}^{m_2} \Gamma(d_j - \delta_j \xi) \prod_{j=1}^{n_2} \Gamma(1 - c_j + \gamma_j \xi)}{\prod_{j=1}^{q_2} \Gamma(1 - d_j + \delta_j \xi) \prod_{j=1}^{p_2} \Gamma(c_j - \gamma_j \xi)},$$

$$\theta_3(\eta) = \frac{\prod_{j=1}^{m_3} \Gamma(f_j - F_j \eta) \prod_{j=1}^{n_3} \Gamma(1 - e_j + E_j \eta)}{\prod_{j=1}^{q_3} \Gamma(1 - f_j + F_j \eta) \prod_{j=1}^{p_3} \Gamma(e_j - E_j \eta)},$$

$x$  and  $y$  are not equal to zero, and an empty product is interpreted as unity  $p_i, q_i, n_i$  and  $m_j$  are non negative integers such that  $p_i \geq n_i \geq 0, q_i \geq 0, q_j \geq m_j \geq 0, (i = 1, 2, 3; j = 2, 3)$ . Also, all the  $A$ 's,  $\alpha$ 's,  $B$ 's,  $\beta$ 's,  $\gamma$ 's,  $\delta$ 's,  $E$ 's, and  $F$ 's are assumed to be positive quantities for standardization purpose.

The contour  $L_1$  is in the  $\xi$ -plane and runs from  $-i\infty$  to  $+i\infty$ , with loops, if necessary, to ensure that the poles of  $\Gamma(d_j - \delta_j \xi)$  ( $j = 1, \dots, m_2$ ) lie to the right, and the poles of  $\Gamma(1 - c_j + \gamma_j \xi)$  ( $j = 1, \dots, n_2$ ),  $\Gamma(1 - a_j + \alpha_j \xi + A_j \eta)$  ( $j = 1, \dots, n_1$ ) to the left of the contour.

The contour  $L_2$  is in the  $\eta$ -plane and runs from  $-i\infty$  to  $+i\infty$ , with loops, if necessary, to ensure that the poles of  $\Gamma(f_j - F_j \eta)$  ( $j = 1, \dots, m_3$ ) lie to the right, and the poles of  $\Gamma(1 - e_j + E_j \eta)$  ( $j = 1, \dots, n_3$ ),  $\Gamma(1 - a_j + \alpha_j \xi + A_j \eta)$  ( $j = 1, \dots, n_1$ ) to the left of the contour.

The generalized H-function of two variables given by (1) is convergent if

$$U = \sum_{j=1}^{n_1} \alpha_j + \sum_{j=1}^{m_1} \beta_j + \sum_{j=1}^{n_2} \gamma_j + \sum_{j=1}^{m_2} \delta_j - \sum_{j=n_1+1}^{p_1} \alpha_j - \sum_{j=m_1+1}^{q_1} \beta_j - \sum_{j=n_2+1}^{p_2} \gamma_j - \sum_{j=m_2+1}^{q_2} \delta_j; \quad (2)$$

$$V = \sum_{j=1}^{n_1} A_j + \sum_{j=1}^{m_1} B_j + \sum_{j=1}^{n_3} E_j + \sum_{j=1}^{m_3} F_j - \sum_{j=n_1+1}^{p_1} A_j - \sum_{j=m_1+1}^{q_1} B_j - \sum_{j=n_3+1}^{p_3} E_j - \sum_{j=m_3+1}^{q_3} F_j, \quad (3)$$

where  $|\arg x| < \frac{1}{2} U\pi, |\arg y| < \frac{1}{2} V\pi$ .

In the present investigation we require the following results:  
Following modified form of the integral [2, p.372, (1)]:

$$\int_0^\pi (\sin x)^{s-1} \cos nx \, dx = \frac{\pi \cos \frac{1}{2} n\pi \Gamma(s)}{2^{s-1} \Gamma\{\frac{1}{2}(s+n+1)\} \Gamma\{\frac{1}{2}(s-n+1)\}}, \quad (4)$$

$\operatorname{Re}(s) > 0$ .

## 2. INTEGRAL:

The integral to be established here is

$$\int_0^\pi (\sin x)^{s-1} \cos nx H \left[ \begin{matrix} z_1 (\sin x)^\lambda \\ z_2 \end{matrix} \right] dx$$

$$= 2^{1-s} \pi \cos \frac{n\pi}{2} H_{p_1, q_1; p_2+1, q_2+2; p_3, q_3}^{m_1, n_1; m_2, n_2+1; m_3, n_3} \left[ \begin{matrix} z_1 2^{-\lambda} \\ z_2 \end{matrix} \right]$$

$$\left| \begin{matrix} (a_j, \alpha_j; A_j)_{1, p_1} : (1-s, \lambda), (c_j, \gamma_j)_{1, p_2} : (e_j, E_j)_{1, p_3} \\ (b_j, \beta_j; B_j)_{1, q_1} : (d_j, \delta_j)_{1, q_2}, (\frac{1}{2} - \frac{s}{2} \pm \frac{n}{2}, \frac{\lambda}{2}) : (f_j, F_j)_{1, q_3} \end{matrix} \right] \quad (5)$$

valid under the condition (1).

**Proof:**

Replace the generalized H-function of two variables by its equivalent contour integral as given in (1), change the order of integration, evaluate the inner integral with the help of (4) and finally interpret it with (1), to get (5).

**3. HEAT CONDUCTION IN A SQUARE PLATE:**

In this section, we consider a problem on heat conduction in a square plate under certain boundary conditions. If a square plate has its faces and its edges  $x = 0$  and  $x = \pi$  ( $0 < y < \pi$ ) insulated, its edges  $y = 0$  and  $y = \pi$  are kept at temperature zero and  $f(x)$  respectively, then its steady temperature  $u(x, y)$  is given by [1, p.125]:

$$u(x, y) = \frac{a_0}{2\pi} y + \sum_{n=1}^{\infty} a_n \frac{\sinh ny}{\cosh nx} \cos nx \quad (6)$$

where

$$a_n = (2/\pi) \int_0^\pi f(x) \cos nx \, dx, \quad n = 0, 1, 2, \dots \quad (7)$$

Now we shall consider the problem of determining  $u(x, y)$ , where

$$u(x, 0) = f(x)$$

$$= (\sin x)^{s-1} H \left[ \begin{matrix} z_1 (\sin x)^\lambda \\ z_2 \end{matrix} \right] \quad (8)$$

**Solution of the Problem:**

Combining (8) and (7) and making the use of the integral (5), we derive

$$a_n = 2^{2-s} \cos \frac{n\pi}{2} H_{p_1, q_1; p_2+1, q_2+2; p_3, q_3}^{m_1, n_1; m_2, n_2+1; m_3, n_3} \left[ \begin{matrix} z_1 2^{-\lambda} \\ z_2 \end{matrix} \right]$$

$$\left| \begin{matrix} (a_j, \alpha_j; A_j)_{1, p_1} : (1-s, \lambda), (c_j, \gamma_j)_{1, p_2} : (e_j, E_j)_{1, p_3} \\ (b_j, \beta_j; B_j)_{1, q_1} : (d_j, \delta_j)_{1, q_2}, (\frac{1}{2} - \frac{s}{2} \pm \frac{n}{2}, \frac{\lambda}{2}) : (f_j, F_j)_{1, q_3} \end{matrix} \right] \quad (9)$$

Putting the value of  $a_n$  from (9) in (6), we get the following required solution of the problem:

$$u(x, y) = \frac{a_0}{2\pi} y + \sum_{n=1}^{\infty} 2^{2-s} \cos \frac{n\pi}{2} \frac{\sinh ny}{\cosh nx} \cos nx$$

$$\times H_{p_1, q_1; p_2+1, q_2+2; p_3, q_3}^{m_1, n_1; m_2, n_2+1; m_3, n_3} \left[ \begin{matrix} z_1 2^{-\lambda} \\ z_2 \\ (a_j, \alpha_j; A_j)_{1, p_1} : (1-s, \lambda), (c_j, \gamma_j)_{1, p_2} : (e_j, E_j)_{1, p_3} \\ (b_j, \beta_j; B_j)_{1, q_1} : (d_j, \delta_j)_{1, q_2}, (\frac{1-s}{2} \pm \frac{n\lambda}{2}) : (f_j, F_j)_{1, q_3} \end{matrix} \right] \quad (10)$$

provided the condition stated with (5) are satisfied.

#### 4. SPECIAL SOLUTIONS:

The importance of the generalized H-function of two variables lies largely from the possibility of expressing by means of the H-symbols a great many of special functions appearing in applied mathematics, physical sciences and statistics. So that each of the solutions given in (10) becomes a master or key solution from which a very large number of solutions can be derived for Meijer's G-function, Generalized Hypergeometric function, Bessel, Legendre, Whittaker functions, their combinations and many other functions.

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## DECOMPOSABILITY OF PROJECTIVE CURVATURE TENSOR IN RECURRENT FINSLER SPACE ( $WR - F_n$ )

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### ABSTRACT

The decomposition of curvature tensor field was studied by K. Takano[1]. The decomposability of curvature tensor in Finsler manifold was studied by Pandey[2]. The purpose of the present chapter is to decompose the projective curvature tensor in recurrent Finsler space and study the properties of conformal decomposition tensor.

### Keywords

Finsler space, projective curvature tensor, recurrent Finsler space.

### 1. INTRODUCTION:

We considered an n-dimensional Finsler space  $F_n$  in which the projective curvature tensor, projective tensor field and deviation tensor field are defined by Rund[3]

$$(1.1) \quad \left\{ \begin{array}{l} \text{(a) } W_{jkh}^i = H_{jkh}^i + \frac{2\delta_j^i}{n+1} H_{[jk]} + \frac{2\dot{x}^i}{n+1} \dot{\partial}_j H_{[kh]} \\ \quad + \frac{\delta_k^i}{n^2-1} (nH_{jh} + H_h + \dot{x}^r \dot{\partial}_j H_{hr}) - \\ \quad \frac{\delta_h^i}{n^2-1} (nH_{jk} + H_{kj} + \dot{x}^r \dot{\partial}_j H_{kr}) \\ \text{(b) } W_{jk}^i = H_{jk}^i + \frac{\dot{x}^i}{n+1} H_{[jk]} \\ \quad + 2 \left\{ \frac{\delta_{[j}^i}{n^2-1} (nH_{k]} - \dot{x}^r H_{k]r}) \right\} \end{array} \right.$$

$$(1.2) \quad W_j^i = H_{jk}^i - H\delta_j^i - \frac{1}{n+1}(\partial_r H_j^r - \partial_j H)\dot{x}^i,$$

respectively. The following relations will be used in our discussion follow from (1.1)(a) and (1.2).

$$(1.3) \quad \begin{cases} a) W_{jkh}^i \dot{x}^j = W_{kh}^i, \\ b) W_{kh}^i \dot{x}^k = W_h^i, \\ c) W_h^i \dot{x}^h = 0. \end{cases}$$

The deviation tensor  $W_k^i$  is homogeneous of second degree in its directional arguments. The Projective tensor  $W_{jk}^i$  is skew-symmetric in its lower indices and projectively homogeneous of degree one in their directional arguments and the projective curvature tensor  $W_{jkh}^i$  is skew-symmetric in its indices  $k$  and  $h$  and is positively homogeneous of degree zero in its directional arguments.

Sinha and Singh[4] have defined that an  $F_n$  is called projective recurrent of the first order if the Berwald's covariant derivative of the projective curvature tensor satisfies

$$(1.4) \quad W_{jkh(l)}^i = V_l W_{jkh}^i,$$

where  $V_l$  is a recurrent vector field. The space equipped with such recurrent vector field and projective curvature tensor is called recurrent Finsler space.

Transvecting (1.4) successively by  $\dot{x}^j$  and  $\dot{x}^k$  and therefore using (1.3)(a) and (1.3)(b), we get

$$(1.5) \quad W_{kh(l)}^i = V_l W_{kh}^i,$$

$$(1.6) \quad W_{h(l)}^i = V_l W_h^i,$$

In view of (1.5) and (1.6). We observe that projective deviation tensor  $W_h^i$  and the projective tensor  $W_{kh}^i$  are recurrent.

The projective curvature tensor satisfied the identity by Sinha, Singh and Tripathi[7]

$$(1.7) \quad W_{hjk(l)}^i + W_{hkl(j)}^i + W_{hlj(k)}^i = 0.$$

Sinha and Singh [5] have also defined that an  $F_n$  is called projective recurrent of second order, If the Weyl's projective curvature tensor satisfies

$$(1.8) \quad W_{jkh(l)(m)}^i = U_{lm} W_{jkh}^i,$$

where  $U_{lm}$  is a recurrence tensor. Transvecting (1.8) successively by  $\dot{x}^j$  and  $\dot{x}^k$ , we get

$$(1.9) \quad W_{kh(l)(m)}^i = U_{lm} W_{kh}^i,$$

and

$$(1.10) \quad W_{h(l)(m)}^i = U_{lm} W_h^i.$$

Accordingly, we can state that projective deviation tensor and projective tensor satisfies the second order recurrent condition, if so is Weyl's curvature tensor.

The recurrent curvature tensor  $H_{jkh}^i$  satisfies the relation Sinha and Singh[6]: Type equation here.

$$(1.11) \quad H_{jkh(l)}^i = \lambda_l H_{jkh}^i$$

where  $V_l$  is recurrence vector. Transvecting (1.11) successively by  $\dot{x}^j$  and  $\dot{x}^k$  and, we get

$$(1.12) \quad H_{kh(l)}^i = \lambda_l H_{kh}^i$$

and

$$(1.13) \quad H_{h(l)}^i = \lambda_l H_h^i,$$

The curvature tensor field of second order satisfies the relations Sinha and Singh[6]:

$$(1.14) \quad H_{jkh(l)(m)}^i = K_{lm} H_{jkh}^i,$$

where  $K_{lm} = \lambda_m + \lambda_{l(m)}$  is the recurrence tensor. Transvecting successively by  $\dot{x}^j$  and  $\dot{x}^k$ , we have

$$(1.15) \quad H_{kh(l)(m)}^i = K_{lm} H_{kh}^i,$$

and

$$(1.16) \quad H_{h(l)(m)}^i = K_{lm} H_h^i,$$

## 2. DECOMPOSITION OF PROJECTIVE CURVATURE TENSOR IN RECURRENT FINSLER SPACE ( $WR - F_n$ ):

Let us consider the projective curvature tensor  $W_{jkh}^i$  in the form

$$(2.1) \quad W_{jkh}^i = Y_j^i B_{kh},$$

where  $Y_j^i$  is non zero tensor and  $B_{kh}$  is skewsymmetric decomposition tensor.

The space equipped with such decomposition of projective curvature tensor with recurrent Finsler space is called decomposition of projective curvature tensor in recurrent Finsler space and we denote it by  $WR - F_n$ .

Differentiating (2.1) covariantly with respect to  $x^l$  in the sense of Berwald's, we get

$$(2.2) \quad W_{jkh(l)}^i = Y_{j(l)}^i B_{kh} + B_{kh(l)} Y_j^i.$$

Using the equation (2.1) in (2.2), we get

$$(2.3) \quad W_{jkh(l)}^i = \beta_l Y_j^i B_{kh} + B_{kh(l)} Y_j^i,$$

where

$$(2.4) \quad Y_{j(l)}^i = \beta_l Y_j^i.$$

From equation (2.1) and equation (2.3), we get

$$(2.5) \quad B_{kh(l)} = (V_l - \beta_l) B_{kh}.$$

Let us assume that  $(V_l \neq \beta_l)$  then the equation (2.5) may be written as

$$(2.6) \quad B_{kh(l)} = \gamma_l B_{kh},$$

where  $\gamma_l = (V_l - \beta_l)$ .

Conversely, If the above equation (2.6) is true then (2.3) yield

$$(2.7) \quad V_l B_{kh} = (\beta_l + \gamma_l) B_{kh}$$

Accordingly, we have the

**Theorem 2.1:** In  $WR - F_n$ , the necessary and sufficient condition for the decomposition tensor  $B_{kh}$  to be recurrent is that the recurrent vector  $V_l$  is not equal to recurrent vector  $\beta_l$ .

Let us assume that the vector  $V_l$  is equal to recurrent vector  $\beta_l$  such that

$$(2.8) \quad V_l = \beta_l$$

In view of equation (2.8), equation (2.5) immediately reduces to

$$(2.9) \quad B_{kh(l)} = 0.$$

Using equation (2.9) in (2.3), we have

$$(2.10) \quad W_{jkh(l)}^i = Y_{j(l)}^i B_{kh}$$

$$\text{or} \quad W_{jkh(l)}^i = \beta_l Y_j^i B_{kh}$$

Adding the expressions obtained by cyclic change of (2.10) with respect to the indices  $k$ ,  $h$  and  $l$ , we have

$$(2.11) \quad W_{jkh(l)}^i + W_{jhl(k)}^i + W_{jlk(h)}^i = Y_j^i (\beta_l B_{kh} + \beta_k B_{hl} + \beta_h B_{lk}).$$

In view of (1.7) equation (2.11) reduces to

$$(2.13) \quad Y_j^i (\beta_l B_{kh} + \beta_k B_{hl} + \beta_h B_{lk}) = 0.$$

Since  $Y_j^i$  is non zero tensor, it implies

$$(2.13) \quad \beta_l B_{kh} + \beta_k B_{hl} + \beta_h B_{lk} = 0$$

$$\text{or} \quad V_l B_{kh} + V_k B_{hl} + V_h B_{lk} = 0$$

Accordingly state:

**Theorem 2.2:** In  $WR - F_n$ , under the decomposition (2.1), if the vector  $V_l$  is equal to  $\beta_l$ , the decomposition tensor satisfies the following identity (2.13).

Differentiating (2.10) covariantly with respect to  $x^m$  in the sense of Berwald's and using (2.9), we get

$$(2.14) \quad W_{jkh(l)(m)}^i = \beta_{l(m)} Y_j^i B_{kh} + \beta_l Y_{j(m)}^i B_{kh}$$

In view of (2.4) the above equation may be written as

$$(2.15) \quad W_{jkh(l)(m)}^i = (\beta_{l(m)} + \beta_l \beta_m) Y_j^i B_{kh},$$

Using equation (1.10) and (2.1), we get

$$(2.16) \quad U_{lm} Y_j^i B_{kh} = (\beta_{l(m)} + \beta_l \beta_m) Y_j^i B_{kh},$$

From (2.16), we have

$$(2.17) \quad U_{lm} = (\beta_{l(m)} + \beta_l \beta_m)$$

Thus we conclude that

**Theorem 2.3** In  $WR - F_n$ , under the decomposition (2.1), if the vector  $V_l$  is equal to  $\beta_l$  for which recurrence vector field  $\beta_l$  satisfies the condition  $\beta_{l(m)} + \beta_l \beta_m \neq 0$ .

Interchanging the indices  $l$  and  $m$  in (2.15) and subtracting the equation thus obtained to (2.15), we have

$$(2.18) \quad \begin{aligned} & W_{jkh(l)(m)}^i - W_{jkh(m)(l)}^i \\ &= (U_{lm} - U_{ml})Y_j^i B_{kh} B_{kh}. \end{aligned}$$

or

$$W_{jkh(l)(m)}^i - W_{jkh(m)(l)}^i = (\beta_{l(m)} - \beta_{m(l)})Y_j^i B_{kh}.$$

Accordingly we state:

**Corollary 2.1:** In  $WR - F_n$ , Under the decomposition (2.1) if the vector  $V_l$  is equal to  $\beta_l$ , the projective curvature tensor satisfies the following identity (2.18).

Differentiating (2.6) covariantly with respect to  $x^m$  in the sense of Berwald's, we get

$$(2.19) \quad \begin{aligned} B_{kh(l)(m)} &= \gamma_{l(m)} B_{kh} + \gamma_l B_{kh(m)} \\ &= (V_{l(m)} - \beta_{l(m)}) B_{kh} + (V_l - \beta_l) B_{kh(m)} \end{aligned}$$

In view of (2.6), the equation (2.19) may be written as

$$(2.20) \quad B_{kh(l)(m)} = (V_{l(m)} - \beta_{l(m)}) B_{kh} + \gamma_m (V_l - \beta_l) B_{kh}$$

or

$$(2.21) \quad B_{kh(l)(m)} = (V_{l(m)} - \beta_{l(m)}) B_{kh} + (V_m - \beta_m)(V_l - \beta_l) B_{kh}$$

$$(2.22) \quad B_{kh(l)(m)} = (V_{l(m)} - \beta_{l(m)} + V_m V_l - V_m \beta_l - V_l \beta_m + \beta_l \beta_m) B_{kh}$$

**Theorem 2.4:** In  $WR - F_n$ , Under the decomposition (2.1), the second order covariant derivative of decomposition tensor  $B_{kh}$  satisfies the relation (2.22).

In view of equation (2.8), equation (2.22) immediately reduces to

$$(2.23) \quad B_{kh(l)(m)} = 0$$

**Corollary 2.2:**  $WR - F_n$ , the second order covariant derivative of decomposition tensor  $B_{kh}$  vanish, If the vector  $V_l$  is equal to  $\beta_l$ .

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## ONE DIMENSIONAL CUTTING STOCK PROBLEM WITH MINIMUM USABLE RESIDUE: A NEW APPROACH

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### ABSTRACT

This paper defines a new technique of optimization of usable residue in One Dimensional Cutting Stock Problem by introducing a method for having minimum leftover to be used as non-standard stock (NSS). The concept behind the usable leftover is very critical in real time application, since the leftover stock to be maintained is difficult. We propose a method that focuses on to leave minimum usable residue, which takes care of warehousing problem up to some extent. Moreover, it has been noticed that the trim loss in some cases is reduced.

**Keywords:** *minimum usable residue, cutting stock problem, leftover stock.*

**AMS Subject Classification:** 90C90; 90C27, 90C10

### 1. INTRODUCTION:

Cutting Stock Problem (CSP) is a problem that deals with operations on stock utilization(cf.[2],[5]). It is a problem that deals with cutting large pieces available in inventory into a set of smaller pieces in order to satisfy the demands. These problems are relevant in the planning of utilization of materials in industries like iron, steel bar factories, glass industries etc., to avoid large amount of waste and minimize the total number of stock length cut and also minimize the cost incurred.

Various methods are developed to solve CSP which are based on pattern-oriented (cf.[8],[15]), item-oriented (cf.[10]) or mixed approach (cf.[12]) and also methods are introduced by Cherri [4] (see also[3]) to present optimal solution which are applicable to small size problems.

Further, there are articles in the literature which shows the reuse of leftover material in the form of pieces(cf.[7],[9]). The two popular methods COLA (cf.[11]) and CUT (cf.[10]) deal with such type of situations in which there is a provision of **reuse** of leftover material. Also later Abuabara[1] modified the model proposed by Gradisar [10](see also[11]) with minimum number of constraints and variables in the model.

In this paper, we propose an efficient resolution to the Cutting Stock Problem in which a heuristic method is developed that defines Cutting Stock Problem with minimum Usable Residue (CSPMUR). Reconsideration on the conventional heuristic method, the CSPMUR characterizes minimum residue which can be discarded and it will not affect the trim loss. In order to handle smaller inventory, we have imposed a condition on our mathematical model such that the residue under consideration should be minimum i.e. within the lower and higher order length. Therefore CSPMUR deals to leave minimum residue which is manageable.

This approach is an extended version of Gradisar's concept on minimum trim loss with usable leftover. This process is feasible when order lengths are small with proportion to stock length such that the sum of the length of all the orders should not exceed the stock length. This can be made relevant for the application to Coronary Stent where large piece of stent have to be cut into small pieces according to the requirement of the Patient. But this concept is not feasible in the case of transmission tower industry where the ratio of stock length (varies from 7m to 14m) to order length is low or medium.

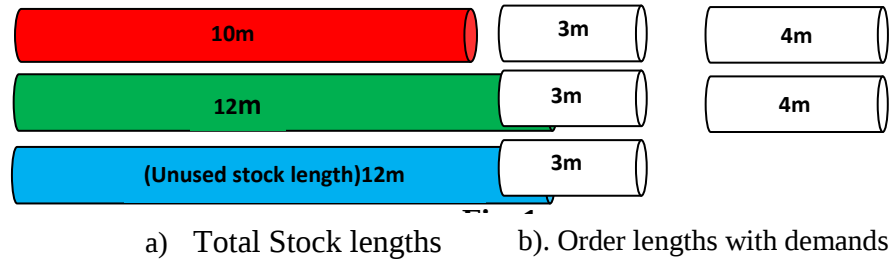
The authors have worked out with the data extracted from [6] and compared with the existing methods and have obtained better results.

## **2. DEFINITION OF THE CUTTINGSTOCK PROBLEM WITH MINIMUMUSABLE RESIDUE:**

In the CSP the scrap is unavoidable which is usually discarded but the question arises if the scrap is big enough (non-standard stock), which cannot be left as useless and if not taken into consideration the consequence is it will affect the functioning of the industry i.e. company will run in loss. Sustenance of then on-standard stock is an extra overhead, since the warehousing is difficult, as it requires space, manpower and maintenance of leftover database. Therefore, we intend to the elucidation of CSP as to minimize the usable scrap and this can be attained by designing the cutting plans in such a way that scraps are minimum that can be ignored or should be larger than the longest order length to go back to the standard stock.

The objective function of conventional CSP is to minimize the waste after cutting the order length from the stock thereby minimizing the waste cost. In this paper, we define the objective function of CSPMUR as to reduce the number of functional (utilizable) pieces left after the order lengths had been cut from the stock length since it is difficult to manage the inventory of these pieces (left over). In accordance with our assertion, we discuss the following illustrative example.

### Example 2.1



### Solution

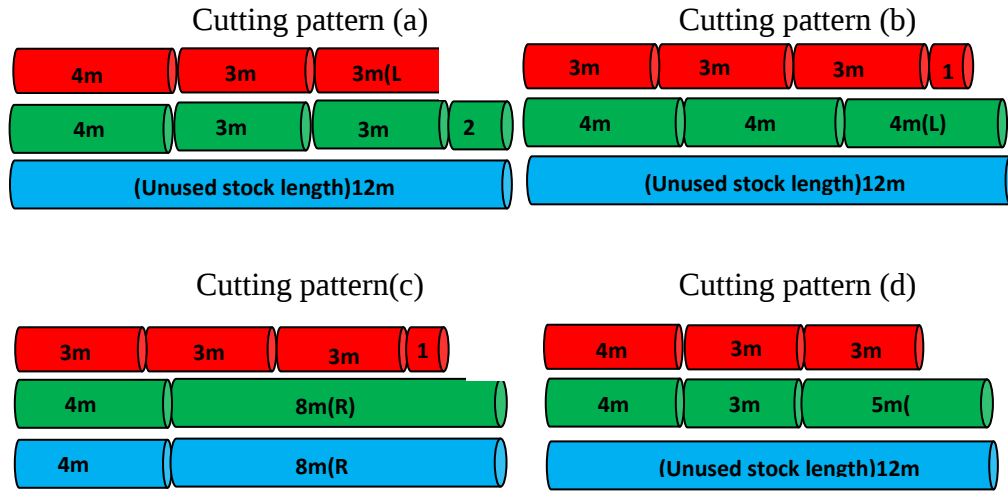


Fig. 2

The stock lengths, order lengths and demands are given in Fig. 1. Various cutting patterns are possible which are represented in Fig.2. Out of options from Fig.2(a-d), Fig.2-d is better when compared with Fig.2(a-c),since it leaves minimum usable residue. According to CSPMUR, the best possible pattern is Fig.2-d, which leaves no trim loss and the residue is large enough which can be merged with the standard stock.

The solution Fig.2a has 3m as usable scrap which is not manageable; similarly usable scrap is also left in solution Fig. 2b and Fig. 2c. Usable scrap is an overhead to the industry, the minimum usable residue can be obtained by focusing on the fact that while cutting plans are determined it should not be focused on a single order length of demand to be cut in multiple number from a single stock rather concentrate the cutting plan of different order length from a single stock to be cut so that scrap can be reduced with minimum usable residue.

### 3. NEW METHOD CSP WITH MINIMUM USABLE RESIDUE INITIATED BASED ON EXISTING METHOD:

These methods are based on the existing objective function i.e. minimizing the trim loss with usable leftover.

- **First Fit Decreasing Algorithm** due to Cherri et.al[6]:  
According to this algorithm the largest stock is initiated to cut the order length until its demand is attained, when it is exhausted, then the next order length is cut from the following second largest stock until the smallest order length is reached.
- **A Generic Greedy Algorithm** due to Sin-Min Lee [14]:  
Many real-world problems are optimization problems in that they attempt to find an optimal solution among many possible proposed solutions.
  - (1) Initialize  $C$  to be the set of proposed solutions
  - (2) Initialize a set  $S = \emptyset$  (the set is to be the optimal solution we are constructing).
  - (3) While  $C \neq \emptyset$  and  $S$  is (still) not a solution do
    - (3.1) select  $x$  from set  $C$  using a greedy strategy
    - (3.2) delete  $x$  from  $C$
    - (3.3) if  $\{x\} \cup S$  is a *feasible* solution, then  
 $S = S \cup \{x\}$  (i.e., add  $x$  to set  $S$ )
  - (4) if  $S$  is a solution then  
Return  $S$
  - (5) else return *failure*

In general, a greedy algorithm is efficient because it makes a sequence of (local) decisions and never backtracks. However, the solution is not always optimal.

These methods as given in the literature aim to minimize the trim loss with no consideration to manage the unavoidable large enough scrap, if discarded will certainly concern the performance of the industry.

In this present paper, authors developed a method of minimizing the non-standard stock(NSS), since it will an overhead for maintaining the data store.

### 4. A NEW APPROACH:

For our cutting plan, we refer the following basic assumptions considered by Dychoff (see [13]) which is basically assortment of large objects. Its individualities are

1. One large object
2. Many identical large objects
3. Different large objects

Besides the above conditions Gradisar in his paper [13] has considered the generalized condition on the available stock length viz. few groups of identical large objects. In view of above considerations, we are now in a position to design our cutting plan which has been described as follows:

Order lengths are arranged in descending order with respect to their lengths. The stock length are considered as G1DCSP or S1DCSP where,

$l_i$  = order length,  $i=1,2,...,n$ .

$d_i$  = required number of pieces of order length  $l_i$

$U$  or  $U_j$  = Stock Length,  $j=1,2,...,p$ .

where  $p$  is the number of diverse standard stock length (cf.[16]) in a adequately big stock.

Choose the appropriate stock length  $U_j$  so that at least one piece of each order length should be cut from  $U_j$ .

$p_{ij}$  = number of pieces of order length  $l_i$  is been cut from stock length  $U_j$

$$\sum_{i=1}^n l_i p_{ij} \leq U_j \quad (4.1)$$

where  $p_{ij} = 1$

Referring equation (4.1) we defined  $\delta_j$  as follows

**Step 1.**

$$\delta_j = U_j - \sum_{i=1}^n l_i p_{ij} \quad ; \text{ for } p_{ij} = 1, j = 1, \dots, p \quad (4.2)$$

The residue left from each stock length should be manageable NSS or non-manageable NSS or minimum NSS which can be discarded for which the objective function is

$$\min \sum_{j=1}^p \delta_j \quad (4.3)$$

(minimizing the usable residue)

subject to

$$\sum_{i=1}^n l_i \leq U_j \quad (4.4)$$

We assume  $p_{ij} = 1$  (4.1)

$$\sum_{i=1}^n p_{ij} = d_i \quad ; i = 1, \dots, n; \quad j = 1, \dots, p \quad (4.5)$$

We check the following:

**Step 2**

- i. If  $\delta_j < l_1$ , then  $\delta_j$  is scrap which shall not be reused
- ii. If  $\delta_j \geq l_1$ , then we shift it to the stock (NSS) which is reusable for further processing.

If (i) holds, then we take up another stock length and proceed as in case of step 1.

If (ii) holds, then we define

$J = \{1, 2, \dots, n\}$

$$\delta_j^1 = \delta_j - \sum_{j \in J} l_j \quad ; p_{ij} = 1 \quad (4.6)$$

where  $J$ (Index set).

We choose the order lengths  $l_j$ 's in (4.6) in such a way that  $\delta_j^1 \geq 0$  i.e., we may not consider some of the order lengths in  $\sum_{j \in J} l_j$ .

Again, we examine  $\delta_j^1$  as in step 2. We continue the process by defining  $\delta_j^n$  till  $\delta_j^n < l_1, i = 1, \dots, n$ .

**Algorithm:**

Step 1: read  $l_i, d_i$  and  $U_j$

Step 2: arrange  $l_i$  and  $U_j$  in descending order.

Step 3:  $\delta_j = U_j - \sum l_i$

If( $\delta_j \leq \min$ ) && ( $\delta_j \geq 0$ )

$\min = \delta_j$

$st_k = U_j$

Step 4: Repeat step 3 for  $U_j \leq U_p$

Step 5:  $p_{ij} = 1 \quad \forall i = 1, \dots, n$

$j++$

$l_i++$

Step 6: repeat step 5  $l_i \leq l_n$

Step 7: if ( $\sum_{i=1}^n p_{ij}$ ) =  $d_i$

$d_i = 0$

Step 8: if  $\delta_j \geq l_i$

$U_j = \delta_j$

$p_{ij}++$  for  $l_i$

if all  $d_i = 0$  then

stop

else

goto step 3.

## 5. COMPARATIVE STUDY OF THE NEW APPROACH WITH FFD AND GENERIC GREEDY ALGORITHM:

In order to compare explicitly our method with FFD and Greedy, we consider **the same data** as analyzed by Gradisar(cf.[6]) in this section .

We now consider the following table of data extracted from [6] with stock length assumed as 3000 cm.

| Item | Length(cm) | Demand |
|------|------------|--------|
| 1    | 250        | 2      |
| 2    | 275        | 2      |
| 3    | 285        | 4      |
| 4    | 525        | 4      |
| 5    | 1380       | 4      |

**Table 1**

### First fit Decreasing (FFD)

In FFD the order lengths and the stock lengths are arranged in a descending order. The order lengths are cut from the stock in multiple factors to satisfy the demand. The longest stock is initiated to cut to satisfy the largest order length, until the demand is exhausted. Then the next order length is cut from the following largest stock until we reach the smallest order.

### Generic Greedy

The various cut patterns are proposed by cutting the order lengths by various stock length out of which the optimal pattern is chosen to solve the 1DCSP.

### Our approach

It deals with the fact that the at least one order length is cut from the stock since the sum of the order length is less than the stock and this process is continued till the scrap left can be discarded.

### Comparative analysis

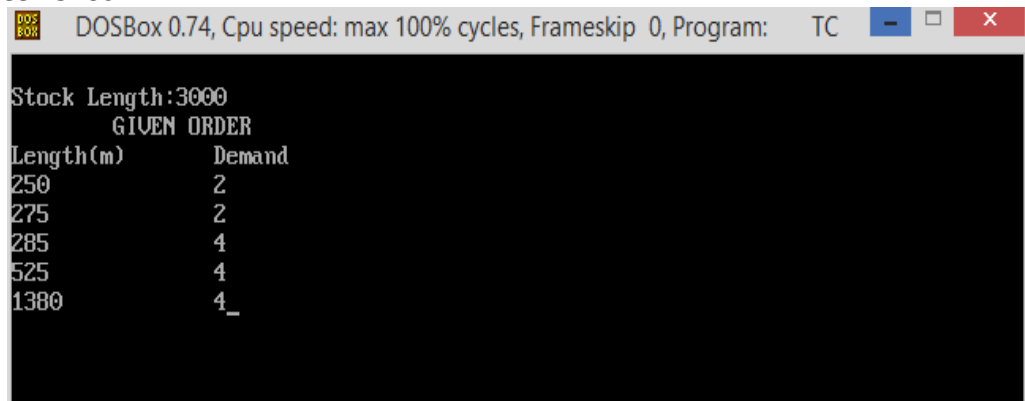
The above methods are applied on data of Table 1 and found that the trim loss computed by the new approach is very less as compared to other two algorithms. Also the scrap left between the smallest order length and the largest order length is almost nil which is the objective of the new approach ie. resolving the problem of maintaining the data warehouse (Table 2).

|                         | Constructive |        |              |
|-------------------------|--------------|--------|--------------|
|                         | FFD          | Greedy | New approach |
| <b>Object cut</b>       | 4            | 4      | 4            |
| <b>Total Length</b>     | 12000        | 12000  | 12000        |
| <b>Total Loss</b>       | 525          | 240    | 4            |
| <b>Total NSS</b>        | 1669         | 1954   | 2190         |
| <b>Avoidable Scrap</b>  | 0            | 0      | 0            |
| <b>Manageable NSS</b>   | 3            | 1      | 2            |
| <b>Unmanageable NSS</b> | 1            | 1      | 0            |

**Table 2**

It has been tested in few problems corresponding to randomly generated data and result was found satisfactory. The above algorithm was computerized in C language to calculate the minimum usable residue of which the screen shots are as below.

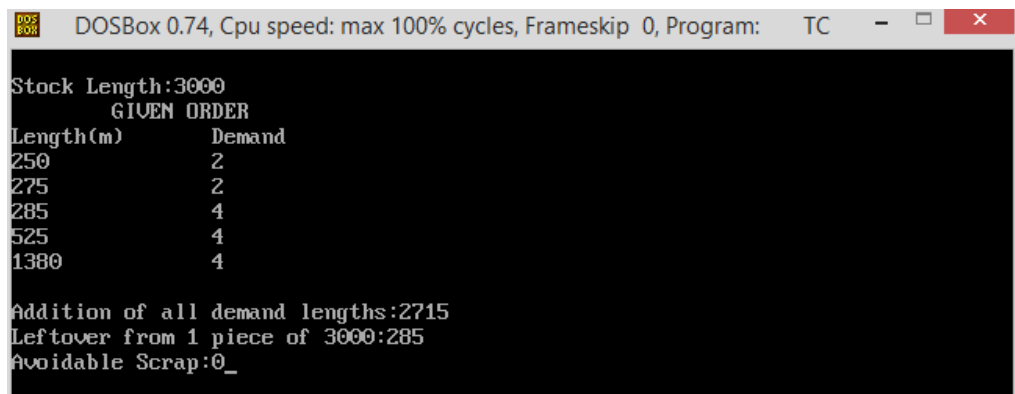
### Screen shot



```

DOSBox 0.74, Cpu speed: max 100% cycles, Frameskip 0, Program: TC
Stock Length:3000
GIVEN ORDER
Length(m)    Demand
250          2
275          2
285          4
525          4
1380         4_
  
```

**Screen Shot 1.** Entering the value of order length, demand and the stock

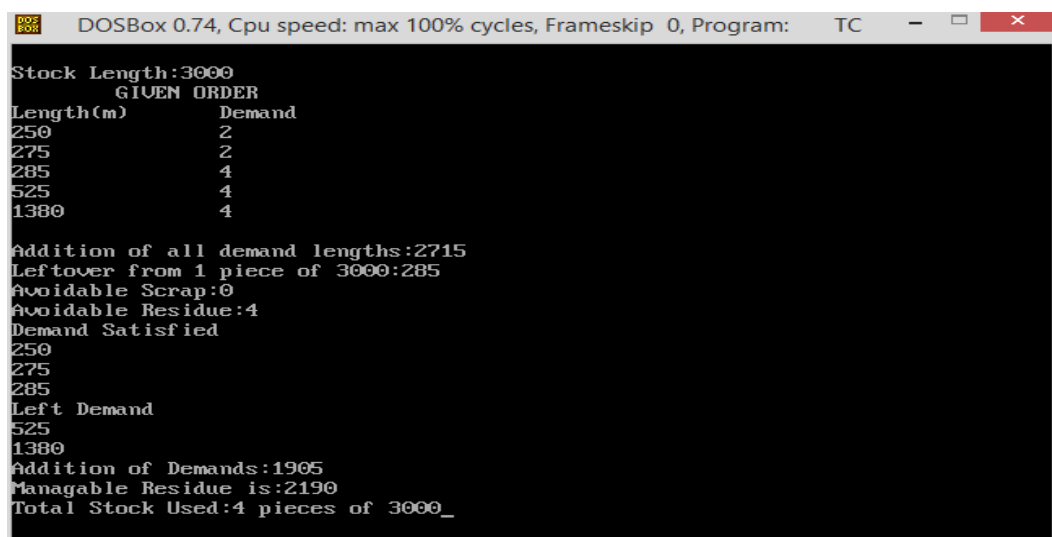


```

DOSBox 0.74, Cpu speed: max 100% cycles, Frameskip 0, Program: TC
Stock Length:3000
GIVEN ORDER
Length(m)    Demand
250          2
275          2
285          4
525          4
1380         4

Addition of all demand lengths:2715
Leftover from 1 piece of 3000:285
Avoidable Scrap:0_
  
```

**Screen Shot 2.** Sum of the order length and leftover after cut pattern

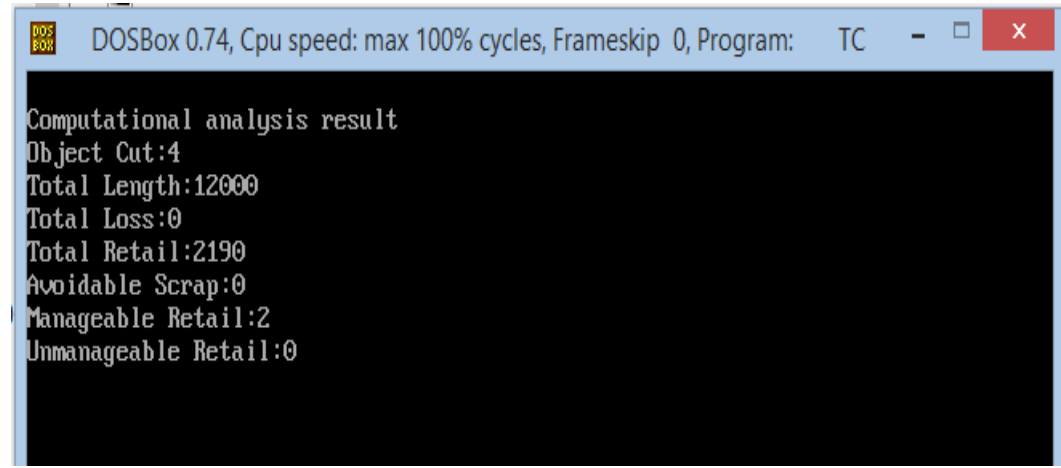


```

DOSBox 0.74, Cpu speed: max 100% cycles, Frameskip 0, Program: TC
Stock Length:3000
GIVEN ORDER
Length(m)    Demand
250          2
275          2
285          4
525          4
1380         4

Addition of all demand lengths:2715
Leftover from 1 piece of 3000:285
Avoidable Scrap:0
Avoidable Residue:4
Demand Satisfied
250
275
285
Left Demand
525
1380
Addition of Demands:1905
Managable Residue is:2190
Total Stock Used:4 pieces of 3000_
  
```

**Screen Shot 3:** Utilization of the stock to satisfy the demand

A screenshot of a DOSBox window. The title bar reads "DOSBox 0.74, Cpu speed: max 100% cycles, Frameskip 0, Program: TC". The window contains a black terminal area with white text displaying the following results:

```
Computational analysis result
Object Cut:4
Total Length:12000
Total Loss:0
Total Retail:2190
Avoidable Scrap:0
Manageable Retail:2
Unmanageable Retail:0
```

**Screen Shot 4:** Computational Analysis Result

### **Conclusion:**

This work characterizes a different technique to resolve the 1D-CSP in some cases, which directs to a better possible resolution with low time complexity. The proposed method is suitable when the ratio of size of stock to order length is sufficiently large. It focuses on the cutting plan that the order lengths are cut from stock in such a way that the usable residue left is minimum.

In this approach we have assumed to cut all the order lengths from the stock which may be observed very rare practically, but the future aspect of research can be to cut few order lengths from the stock to reduce the operating cost which will be more feasible from implicational point of view.

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## **CONSTRUCTING APPROXIMATIONS TO BIVARIATE PIECEWISE-SMOOTH FUNCTIONS**

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### **ABSTRACT**

This paper demonstrates that the space of piecewise smooth functions can be well approximated by the space of functions defined by a set of simple (non-linear) operations on smooth uniform splines. The examples include bivariate functions with jump discontinuities or normal discontinuities across curves, and even across more involved geometries such as a 3-corner. The given data may be uniform or non-uniform, and noisy, and the approximation procedure involves non-linear least-squares minimization. Also included is a basic approximation theorem for functions with jump discontinuity across a smooth curve.

### **1. INTRODUCTION:**

High-quality approximations of piecewise-smooth functions from a discrete set of function values is a challenging problem with applications in image processing and geometric modeling. The univariate problem has been studied by several research groups, and satisfactory solutions can be found in the works of: Harten [6], Arandifa et al.[1], Archibald et al.[2, 3], Lipman et al.[9]. However, the 2D problem is still far from being solved, and the 1D methods are not easily adapted to the real 2D case. Furthermore, even the 1D problem is not easily solved in presence of noisy data. In the 1D problem we are given values of a piecewise smooth function, with or without noise, and the challenge is to approximate the location of the 'singular points' which separate one smooth part of the function from the other, and to also reconstruct the smooth parts. In the 2D case a piecewise smooth function on a domain  $D$  is defined by a partition of the domain into segments separated by boundary curves (smooth or non-smooth), and the function is smooth in the interior of each segment. By the term smooth we mean that the derivatives (up to a certain order) of the function are bounded. Of course, the function and/or its derivatives may be discontinuous across a boundary curve between segments. Given data acquired from such an underlying piecewise smooth function, the challenge here is to approximate the separating curves (the singularity curves), and to reconstruct the smooth parts. Note that apart from noise in the function values, there may also be a 'noise' in the location of the separating curves (as demonstrated in Section 3.2).

The problem of approximating piecewise-smooth functions is a model problem for image processing algorithms, and some sophisticated classes of wavelets and frames have been designed to approximate such functions. For example, see Candes and Donoho [5]. A method for approximation piecewise smooth functions would also be useful for the reconstruction of surfaces in CAGD or in 3D computer graphics, e.g., via the moving least-squares framework.

It is well established now that only non-linear methods may achieve optimal approximation results in 'non-smooth' spaces, e.g., see Binev et al.[12]. In this paper we are going back to using the 'good old' splines with uniform knots as our basis functions for the approximation, but we add to the game some (simple) non-linear operations on the space of splines. In fact, all the non-linearity used here can be expressed by the sign operation. We remark that the choice of the spline basis is not essential here, and other basis functions may be utilized within the same framework.

We present the idea and the proposed approximation algorithms through a series of illustrative examples. Building from derivative discontinuity in the univariate case, we move into normal discontinuity and jump discontinuity across curves in the bivariate case, with some non-trivial topologies of the singularity curves. We shall also present a basic approximation result for the case of jump discontinuity across a smooth curve. Altogether, we present a simple, yet powerful approach to piecewise smooth approximation. The suggested method seems to be quite robust to noisy data, and even the univariate version is interesting in this respect. Open issues, as the development of efficient algorithms and further approximation analysis are left for future research.

## 2. NON-SMOOTH UNIVARIATE APPROXIMATIONS:

To demonstrate the main idea, we start with the univariate problem: Assume we know that our underlying univariate function  $f$  is continuous,  $f \in C[a, b]$ , and that it has one point of discontinuity in its first derivative in  $s \in [a, b]$ , and that  $f'(s^-) > f'(s^+)$ . Then it makes sense to look for two smooth functions  $g^{[r]}$  and  $g^{[l]}$ , where  $g^{[l]}$  approximates  $f$  on the left segment  $[a, s]$ , and  $g^{[r]}$  approximates  $f$  on the right segment  $[s, b]$ , and such that

$$(2.1) \quad f(x) = \min(g^{[l]}(x), g^{[r]}(x)), \forall x \in [a, b].$$

$g^{[r]}$  may be viewed as a smooth extension of  $f|_{[a, s]}$  to the whole interval  $[a, b]$ , and  $g^{[l]}$  as a smooth extension of  $f|_{[s, b]}$  to  $[a, b]$ . It is clear that there are many pairs of smooth functions  $g^{[r]}$  and  $g^{[l]}$  which satisfy the above relation. Therefore, one may suspect that the problem of finding such a pair is ill-conditioned. Let us check this by trying a specific algorithm for solving this problem, and check it on few examples. It becomes clear from these examples that the approximations by  $g^{[r]}$  and  $g^{[l]}$  are well defined in the relevant intervals, i.e.,  $g^{[r]}$  in  $(s, b]$  and  $g^{[l]}$  in  $[a, s)$ . To approximate the functions  $g^{[r]}$  and  $g^{[l]}$  we use cubic spline basis functions, with equidistant knots  $t_j = a + (j - 1)\delta$ ,  $j = 1, \dots, k$ ,  $\delta = (b - a)/k$ .

Assuming we are given data  $\{f(x_i)|x_i \in X \subset [a, b]\}$ , we look for  $g^{[r]}$  and  $g^{[\ell]}$  such that

$$(2.2) \quad F_1(p) = \sum_{x_i \in X} [f(x_i) - \min(g^{[\ell]}(x_i), g^{[r]}(x_i))]^2 \rightarrow \text{minimum}.$$

Here  $p$  stands for the set of parameters used in the representation of the unknown functions  $g^{[r]}$  and  $g^{[\ell]}$ . We use the convenient representation

$$(2.3) \quad g^{[r]}(x) = \sum_{j=1}^k \alpha_j B_j(x), \quad g^{[\ell]}(x) = \sum_{j=1}^k \beta_j B_j(x).$$

$\{B_j\}_{j=1}^k$  are, for example, the basis functions for cubic spline interpolant with the not-a-knot end conditions, satisfying  $B_j(t_i) = \delta_{i,j}$ . Hence, in (2.2)  $p$  stands for the unknown splines' coefficients,  $p = \{\alpha_j\}_{j=1}^k \cup \{\beta_j\}_{j=1}^k$ .

In Figure 1 and in Figure 2 we see the results of reconstructing piecewise smooth functions from exact data and from noisy data. In both cases  $X = \{-3 : 0.02 : 3\}$  and  $\{t_j\} = \{-3 : 1.5 : 3\}$ . The solution of the optimization problem (2.2) is depicted in a bold line. The underlying function  $f$  is generated as  $f(x) = \min(f^{[\ell]}(x), f^{[r]}(x))$ , and the graphs of these two generating functions are depicted by dashed lines. The fine continuous lines in the figures represent the functions  $g^{[r]}$  and  $g^{[\ell]}$ , which, as we see in those graphs, approximate  $f^{[r]}$  and  $f^{[\ell]}$  accordingly, and the approximation is good only in the appropriate regions. Here  $k = 5$ , and thus we have 10 unknown parameters to solve for. The optimization has been performed using a differential evolution procedure, using the data values at the points  $\{t_j\}$  as the starting points of the iterations for both  $\{\alpha_j\}$  and  $\{\beta_j\}$ .

*Remark 2.1.* An alternative representation of  $f$  in (2.1) is

$$(2.4) \quad f(x) = g^{[\ell]}(x) - (g^{[\ell]}(x) - g^{[r]}(x))_+, \quad x \in [a, b],$$

where

$$(2.5) \quad (t)_+ = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}.$$

Hence, we can replace the cost functional (2.2) by

$$(2.6) \quad F_2(p) = \sum_{x_i \in X} [f(x_i) - (g_1(x_i) - (g_2(x_i))_+)]^2.$$

Here  $p$  stands for the set of parameters in the representation of the unknown spline functions  $g_1$  and  $g_2$ , with the advantage that here only one unknown spline function,  $g_2$ , influences the functional in a non-linear manner. We shall further discuss such semi-linear cases in the bivariate case.

**2.1. The case  $f'(s^-) < f'(s^+)$  and more.** Obviously, in this case we should replace the *min* operation within (2.2) by a *max* operation. In case we have two break points  $s_1$  and  $s_2$  in  $[a, b]$ , e.g. with  $f'(s_1^-) < f'(s_1^+)$  and  $f'(s_2^-) > f'(s_2^+)$ , then we may look for three unknown spline functions,  $g_1, g_2, g_3$ , such that  $\min(g_1, \max(g_2, g_3))$  approximates the data in the least-squares sense, and so on. To avoid high complexity we suggest to subdivide

$[a, b]$  into intervals, partially overlapping, each containing at most one break point, and to blend the individual local approximations into a global one over  $a, b$ . We shall further discuss and demonstrate this strategy in the 2D case.

The problem of approximating piecewise smooth univariate data has been investigated by many authors. A prominent approach to the problem is the so-called essentially non-oscillatory (ENO) and subcell resolution (SR) schemes introduced by Harten [6]. The ENO scheme constructs a piecewise-polynomial interpolant on a uniform grid which, loosely speaking, uses the smoothest consecutive data points in the vicinity of each data cell. The SR technique approximate the singularity location by intersecting two polynomials each from another side of the suspected singularity cell. In the spirit of ENO-SR many interesting works have been written using this simple but powerful idea. Recently, Arandifa *et al.*[1] gave a rigorous treatment to a variation of the technique, proving the expected approximation power on piecewise-smooth data. Archibald *et al.* [2, 3] have further improved the ENO idea by introducing polynomial annihilation techniques for locating the cell which contains the singularity. A recent paper by Lipman *et al.*[9] is using quasi-interpolation operators for this problem. Yet, the extension of the univariate methods to the 2D case is not obvious and is not simple. In [1], after locating a interval of possible singularity using ENO [6], two polynomial approximations are defined, each one approximating the data on one side of the singularity, and their intersection is used to approximate the singularity location. The method suggested here is similar, since we also look for two different approximations related to the two sides of a singularity. However, the least-squares optimization approach enables natural extension to interesting cases in the bivariate case. The singularity localization is integrated within the approximation procedure, and thus it is less sensitive to noise. In the next section we hope to convince that the simple idea represented in Section 2 is has the potential of solving some non-trivial bivariate approximation problems.

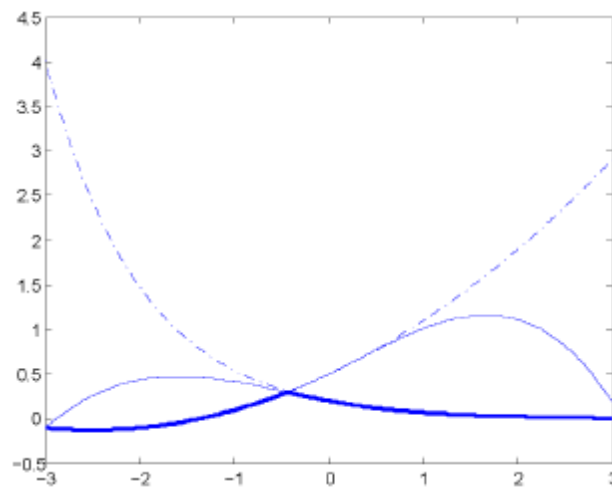


FIGURE 1. A univariate example - No noise

### 3. NON-SMOOTH BIVARIATE APPROXIMATIONS:

As demonstrated in the 1D case, the non-linear space of functions defined by uniform splines, together with the simple operations min and max, may be used to approximate univariate piecewise smooth continuous functions. In the bivariate case we consider functions with derivative discontinuities or jump discontinuities across curves. The objectives of this section are fourfold:

- (1) To exhibit a range of piecewise smooth bivariate functions which can be represented by simple non-linear operations (as min and max) on smooth functions.
- (2) To suggest some non-linear least-squares approximation procedures for the approximation of piecewise smooth bivariate functions.
- (3) To present interesting examples of approximating piecewise smooth bivariate functions, given noisy data.
- (4) To provide a basic approximation result.

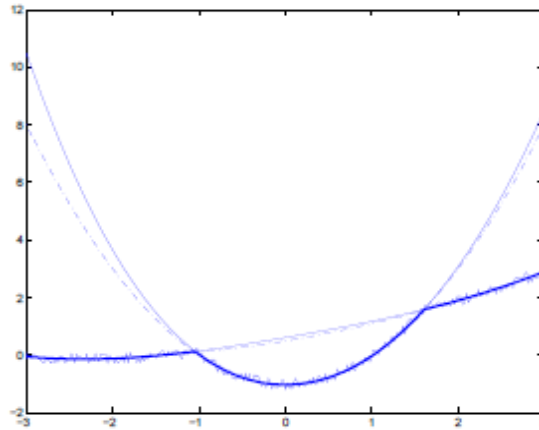


FIGURE 2. A univariate example - Reconstruction in presence of noise

**3.1. Normals' Discontinuity across curves - Problem A.** We start with a numerical demonstration of a direct extension of the univariate approach to the approximation of continuous piecewise smooth bivariate functions. Recalling the 1D discussion, the choice of a *min* or a *max* operation depends on the sign of  $f'(s^+) - f'(s^-)$ . In the 2D case we refer to an analogous condition involving the slopes of the graph along the singularity curves. A discontinuity (singularity) of the normals of a bivariate function  $f$  is said to be convex along a curve  $\gamma$  if the exterior angle of the graph of  $f$  at every point along the curve is  $< \Pi$  (e.g., see Figure 6), and it is considered to be concave if the exterior angles are  $> \Pi$ . In a neighborhood of a concave singularity (discontinuity) curve the function may be described as the minimum between two (or more) smooth functions, and near a convex singularity curve the function may be defined as the maximum of two or more smooth functions. Let us consider the following noisy data,  $\{f(x_i)\}_{x_i \in X}$ , taken from a function with convex singularities. For the numerical experiment we took  $X$  as the set of data points on a square

grid of mesh size  $h = 0.125$  in  $D = [-2, 2] \times [-2, 5]$ , and the given noisy data is shown in Figure 3. In this case the function has a '3-corner' type singularity, where  $f$  has convex singularity along three curves meeting at a point. Therefore, we look for three spline functions,  $g_1, g_2, g_3$  so that

$$(3.1) \quad f(x) \simeq \max(g_1(x), g_2(x), g_3(x)),$$

where  $g_1, g_2, g_3$  solve the non-linear least-squares problem:

$$(3.2) \quad F_A(p) = \sum_{x_i \in X} [f(x_i) - \max(g_1(x_i), g_2(x_i), g_3(x_i))]^2 \rightarrow \text{minimum}.$$

Within this example we would also like to show how to blend two non-smooth approximations. Therefore, we consider the approximation problem on two partially overlapping sub-domains of  $D$ ,  $D_1 = [-2, 2] \times [-2, 2] \subset D$  and  $D_2 = [-2, 2] \times [1, 5] \subset D$ . After solving the approximation problem separately on each sub-domain, the two approximations will be blended into a global one. On each sub-domain the unknown functions  $\{g_i\}_{i=1}^3$  are chosen to be cubic spline functions with a square grid of knots of grid size  $\delta = 2$ . Here again the triplet of functions  $g_1, g_2, g_3$  which solve the minimization problem (3.2) is not unique. However, it turns out that the approximation to  $f$  is well defined by (3.2). I.e., the parts of  $\{g_i\}_{i=1}^3$  which are relevant to  $\max(g_1, g_2, g_3)$  are well defined.

Let us first consider the approximation on the sub-domain  $D_1 = [-2, 2] \times [-2, 2] \subset D$ . For the particular data shown on the left plot in Figure 4, the solution of (3.2) yields the piecewise smooth approximation depicted on the right plot. In this plot we see the full graphs of the three functions  $\{g_i\}_{i=1}^3$  (for this sub-domain), while the approximation is only the upper part (the maximal values) of these graphs. The solution of the optimization problem (3.2) has been found using a differential evolution procedure [7]. As an initial guess for the three unknown functions we took, as in the univariate case, the spline function which approximates the data over the whole domain  $D_1$ . Next, we look for the approximation on  $D_2 = [-2, 2] \times [1, 5] \subset D$ , which partially overlaps  $D_1$ . The relevant data and the resulting approximation are shown in Figure 5.

In order to achieve an approximation over the whole domain  $D = [-2, 2] \times [-2, 5]$ , we now explain how to blend the two approximations defined on  $D_1$  and on  $D_2$ . The singularity curves of the two approximations do not necessarily overlap on  $D_1 \cap D_2$ . Therefore, a direct blending of the two approximations will not provide a smooth transition of the singularity curve. The appropriate blending should be done between the corresponding spline functions generating these singularity curves. On each sub-domain the approximation is defined by another triplet of splines  $\{g_i\}_{i=1}^3$ . For the approximation over  $D_2$  only two of the splines are active in the final  $\max$  operation, and the graph of the third spline is below the maximum of the other two. To prepare for the blending step we have to match appropriate pairs of both triplets, and this can easily be done by proximity over the blending zone  $D_1 \cap D_2$ . The final approximation over  $D$  is defined by  $\max(\tilde{g}_1, \tilde{g}_2, \tilde{g}_3)$ , where  $\{\tilde{g}_i\}_{i=1}^3$  are defined by blending the appropriate pairs, using the simplest  $C^1$  blending function. The resulting blended approximation over  $D$ , to the data given in Figure 3, is displayed in Figure 6.

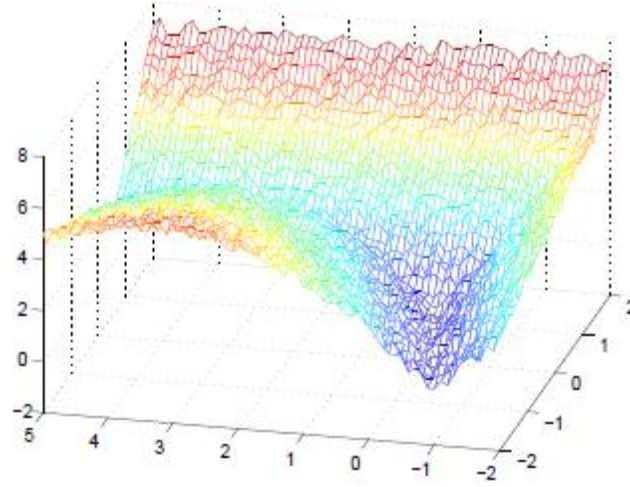


FIGURE 3. The noisy data over  $[-2;2] \times [-2;5]$

**3.2. Jump Discontinuity across a curve - Problem B.** Another interesting problem in bivariate approximation is the approximation of a function with a discontinuity across a curve. Consider the case of a function defined over a domain  $D$ , with a discontinuity across a (simple) curve  $\gamma$ , separating  $D$  into two sub-domains  $D_+$  and  $D_-$ . We assume that  $f|_{D_+}$  and  $f|_{D_-}$  are smooth on  $D_+$  and  $D_-$  respectively. Such problems, and especially

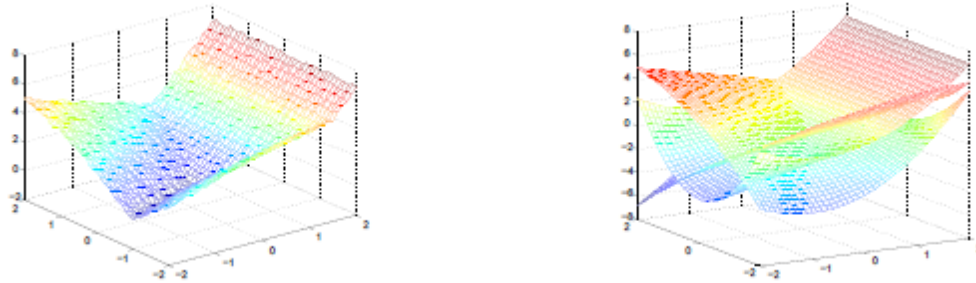


FIGURE 4. The noisy data and the 3-corner approximation over  $D_1$

the problem of approximating  $\gamma$ , appear in image segmentation. Efficient algorithms for constructing  $\gamma$ , which are useful even for more involved data, are the method of snakes, or active contours, and the level-set method. The method of snakes, introduced in [8], iteratively finds contours that approach the contour  $\gamma$  separating two distinctive regions in an image, with applications to shape modelling [10]. The level-set method, first suggested in [11], is also an iterative method for approximating  $\gamma$ , using a variational formulation for minimizing appropriate energy functionals. Recently a variational spline level-set approach has been suggested in [4]. Here, the focus is on simultaneously approximating the curve  $\gamma$  and the function on  $D_+$  and  $D_-$ . This goal is reflected in the cost functional used below, and, as demonstrated in Section 3.5, we can also handle non-simple topologies of  $\gamma$ , such as a 3-corner. The following procedure for treating a jump singularity comes as a natural extension of the framework for approximating a continuous function with derivative discontinuity, as suggested in Section 3.2:

Again, we look for three spline functions,  $g_\gamma$ ,  $g_+$  and  $g_-$ , such that the zero level set  $\tilde{\gamma}$  of  $g_\gamma$  approximates the singularity curve  $\gamma$ ,  $g_+$  approximates  $f$  on  $D_+$ , and  $g_-$  approximates  $f$  on  $D_-$ . Formally, we would like to minimize the following objective function:

$$(3.3) \quad F_B(p) = \sum_{g_\gamma(x_i) > 0} [f(x_i) - g_+(x_i)]^2 + \sum_{g_\gamma(x_i) < 0} [f(x_i) - g_-(x_i)]^2 \rightarrow \text{minimum}.$$

Note that the non-linearity of the minimization problem here, which we denote Problem B, is due to the non-linear operation of *sign* checking. This approximation problem may seem to be more complicated than Problem A of the previous section, but actually it is

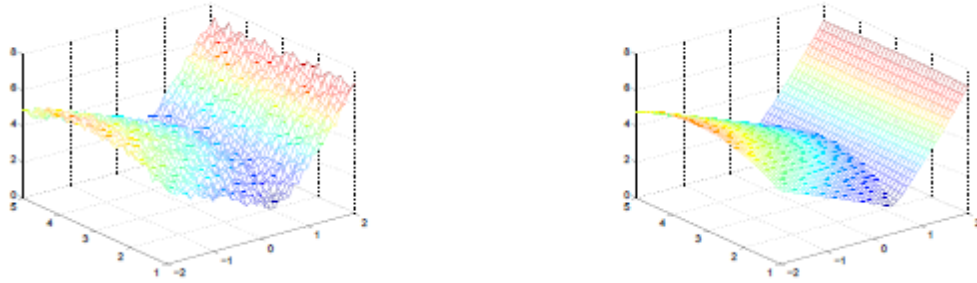


FIGURE 5. The noisy data and the approximation over  $D_2$

somewhat simpler. While in problem A the unknown coefficients of all the three splines appear in a non-linear form in the objective function  $F_A$  (due to the *max* operation), here only the coefficients of  $g_\gamma$  influence the value of  $F_B$  in a non-linear manner. This is due to the observation that once  $g_\gamma$  is known, the functions  $g_+$  and  $g_-$  which minimize  $F_B$  are defined via a linear system of equations. In view of this observation, and for reasons which will be clarified below, we use a slight variation of the optimization problem. Namely, we look for a function  $g_\gamma$  which minimizes  $F_B$ , where  $g_+$  and  $g_-$  are defined by the (linear) least-squares problem:

$$(3.4) \quad \tilde{F}_B(p) = \sum_{x_i \in X_+^h} [f(x_i) - g_+(x_i)]^2 + \sum_{x_i \in X_-^h} [f(x_i) - g_-(x_i)]^2 \rightarrow \text{minimum},$$

where  $\tilde{\gamma}$  denotes the zero level set of  $g_\gamma$ ,  $h$  is the 'mesh size' in the data set  $X$ , and

$$X_+^h = \{x_i \mid g_\gamma(x_i) > 0, \text{dist}(x_i, \tilde{\gamma}) > h\},$$

$$X_-^h = \{x_i \mid g_\gamma(x_i) < 0, \text{dist}(x_i, \tilde{\gamma}) > h\}.$$

For non-noisy data we would like to achieve an  $O(h^4)$  approximation order to  $f|_{D_+}$  and  $f|_{D_-}$ , on  $D_+$  and  $D_-$  respectively. This can be obtained by using proper boundary conditions in the computation of  $g_+$  and  $g_-$ , e.g., by extending the data by local polynomial approximations. We thus consider a third version of the least-squares problem for  $g_+$  and  $g_-$ :

$$(3.5) \quad \tilde{F}_B(p) = \sum_{\hat{X}_+^h} [\hat{f}_+(x_i) - g_+(x_i)]^2 + \sum_{\hat{X}_-^h} [\hat{f}_-(x_i) - g_-(x_i)]^2 \rightarrow \text{minimum}.$$

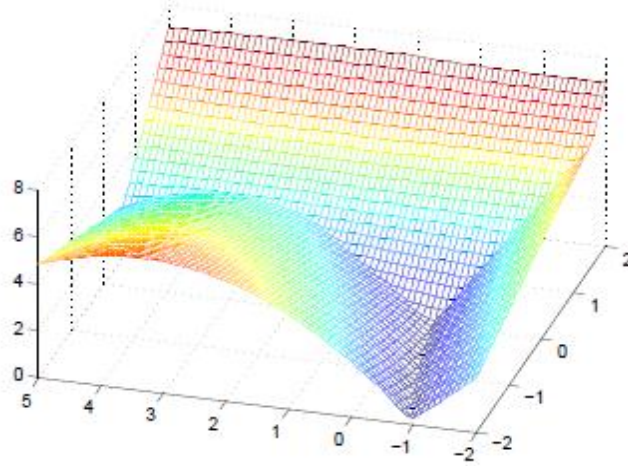


FIGURE 6. The blended approximation over  $[-2;2] \times [-2;2]$

In (3.5)  $\hat{X}_+^h = X \setminus X_-^h$  and  $\hat{X}^h = X \setminus X_+^h$ ,  $\hat{f}_+(x_i)$  is the given data  $f(x_i)$  on  $X_+^h$  and the extension of this data into  $\hat{X}_+^h \setminus X_+^h$ , and  $\hat{f}_-(x_i)$  is the given data  $f(x_i)$  on  $X_-^h$  and the extension of this data on  $\hat{X}_-^h \setminus X_-^h$ . The extension operator should be exact for cubic polynomials.

*Remark 3.1.* Since  $g_\gamma$  may be defined up to a multiplying factor, we may restrict its unknown coefficients to lie in a compact bounded box, and thus the existence of a global minimizer in (3.3)-(3.5) is ensured.

Let us now describe a numerical experiment based upon the above framework. The function we would like to approximate is defined on  $D = [-3, 3]^2$ , and it has a jump discontinuity across a sinusoidal shaped curve. We may consider two types of noisy data; The first includes noise in the data values, and the second includes noise in the location of the singularity curve  $\gamma$ . The three unknown functions  $g_\gamma, g_+, g_-$  are again cubic spline functions with a square grid of knots of grid size  $\delta = 2$ . However, the unknown parameters  $p$  in  $F_B$  are just the coefficients of  $g_\gamma$ . The other two spline functions are computed within the evaluation procedure of  $F_B$  by solving the linear system of equations for their coefficients, i.e., the system defined by the least-squares problem (3.4). The noisy data of the second type (noise in the location of  $\gamma$ ), and the resulting approximation obtained by minimizing (3.3), are displayed in Figures 7 and 8.

For a function with a more involved shape of singularity curve we would suggest to subdivide the domain into patches, partially overlapping, and then blend the approximations over the individual patches into a global approximation. As in the blending suggested for Problem A, the blending of two approximations to jump discontinuities over partially overlapping patches  $D_1$  and  $D_2$  should be performed on the functions  $g_\gamma, g_+, g_-$  which generate the approximations on the different patches. Here one should take care of the fact that the function  $g_\gamma$  is no uniquely defined by the optimization problem (3.3). Let us denote by  $g_{\gamma,1}$  and  $g_{\gamma,2}$  the functions generating the singularity curve on  $D_1$  and  $D_2$  respectively. In order to achieve a nice blending of the two curves we suggest to scale one of the two functions, say  $g_{\gamma,1}$ , so that  $\alpha \cdot g_{\gamma,1} \simeq g_{\gamma,2}$  on  $D_1 \cap D_2$ . In fact, it is important to match the two functions only on that part of  $D_1 \cap D_2$  which is close to the zero curves defined by  $g_{\gamma,1}$  and  $g_{\gamma,2}$ .

**3.3. Problem B - Approximation Analysis.** The approximation problem is as follows: Consider a piecewise smooth function  $f$  defined over a domain  $D$ , with a discontinuity across a simple, smooth curve  $\gamma$ , separating  $D$  into two open sub-domains  $D_+$  and  $D_-$ . We assume that  $f|_{D_+}$  and  $f|_{D_-}$  are smooth, with bounded derivatives of order four on  $D_+$  and  $D_-$  respectively, and so is the curve  $\gamma$ . Let  $X \subset D$  be a grid of data points of grid size  $h$ , and let us consider the approximations for Problem B using bi-cubic spline functions with knots on a grid of size  $\delta = mh$  ( $m > 3$ ). The classical result on approximation by least-squares by cubic splines implies an  $O(h^4)$  approximation order to a function with bounded derivatives of order four (provided there are enough data points for a well-posed solution). On the other hand, even in the univariate case, the location of a jump discontinuity in a piecewise smooth function is inherently up to an  $O(h)$  error. Therefore, the best we can expect from a good approximation procedure for  $f$  such as above is the following:

**Theorem 3.2.** *Consider Problem B on  $D$  and let  $g_\gamma$  be a bi-cubic spline function (with knots' grid size  $\delta = mh$ ) which gives a local minimum to (3.3), with  $g_+$  and  $g_-$  defined by minimizing (3.5). Denote the segmentation defined by  $g_\gamma$  by  $G_+ = \{g_\gamma(x_i) > 0, x_i \in X\}$  and  $G_- = \{x_i | g_\gamma(x_i) < 0, x_i \in X\}$ . For  $C > 0$ , and for  $h$  small enough, there exists such local minimizer  $g_\gamma$  such that if  $x_i \in G_+ \cap D_-$  or  $x_i \in G_- \cap D_+$  then  $\text{dist}(x_i, \gamma) < Ch^3$ .*

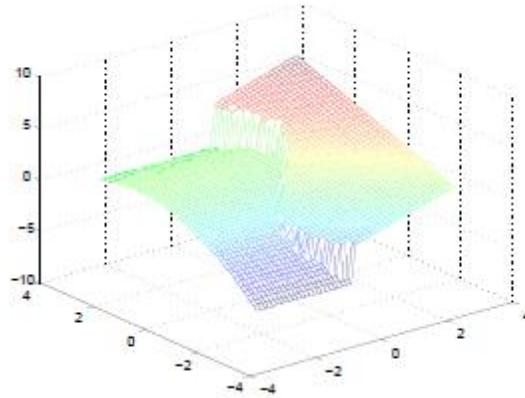


FIGURE 7. Discontinuity across a noisy curve  $[0,3;3] \times [0,3;3]$

*Proof.* The Theorem says that the zero level set of  $g_\gamma, \tilde{\gamma}$ , separates well the data set  $X$  into the two parts, and only data points which are very close to  $\gamma$  may appear in the wrong segment. In order to prove this result we first observe that the curve  $\gamma$  can be approximated by the zero level set of bi-cubic splines with approximation error  $< C_1 h^4$ . One such spline would be  $s_\gamma$ , the approximation to the signed distance function related to the curve  $\gamma$ . Fixing  $g_\gamma = s_\gamma$  determines  $g_+$  and  $g_-$  which minimize  $\tilde{F}_B$  for this  $g_\gamma$ , and we denote the corresponding value  $F_B[s_\gamma]$ . We note that the contribution to the value of  $F_B$  is  $o(h^4)$  (as  $h \rightarrow 0$ ) from a point which fall on the right side of  $\tilde{\gamma}$ , and it is  $O(1)$  from a point on the wrong side of  $\tilde{\gamma}$ . For a small enough  $h$ , only a small number of points  $x_i \in X$  will fall in the wrong side of  $\tilde{\gamma}$ , and any choice of  $g_\gamma$  which induces more points in the wrong side will induce a larger value of  $F_B$ . Obviously, the minimizing solution induces a value

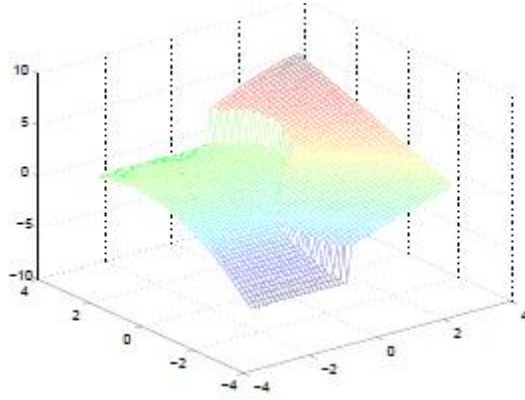


FIGURE 8. The approximation using noisy curve data

$F_B \leq F_B[s_\gamma]$ , and this can be achieved only by reducing the set of 'wrong side' points. Since  $g_\gamma = s_\gamma$  already defines an  $O(h^4)$  separation approximation, only points which are at distance  $O(h^4)$  from  $\gamma$  may stay on the wrong side in the local minimizer which evolves by a continuous change of  $s_\gamma$  which reduces  $F_B$ .  $\square$

**Corollary 3.3.** *If the least-squares problems defining  $g_+$  and  $g_-$  by (3.4) are well-posed, we get*

$$(3.6) \quad \|f - g_+\|_{\infty, D_+} \leq C_2 h^4,$$

$$(3.7) \quad \|f - g_-\|_{\infty, D_-} \leq C_3 h^4.$$

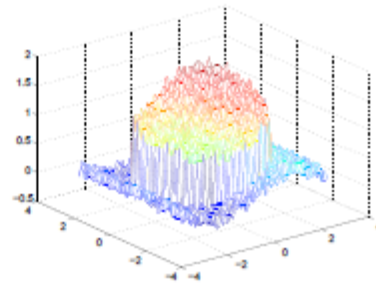
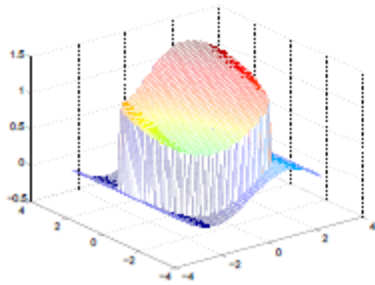


FIGURE 9. The underlying function and its 1st type noisy data

**Remark 3.4.** The above well-posed condition can be checked while computing  $g_+$  and  $g_-$ . Also, an  $O(h^4)$  approximation order can be obtained by using proper boundary conditions in the computation of  $g_+$  and  $g_-$ , e.g., by extending the data by local polynomial approximations, as suggested in (3.5).

**Remark 3.5.** The need to restrict the set of data points defining  $g_+$  and  $g_-$  in (3.4) emerged in view of the condition needed for the proof of Theorem 3.2. As shown in the numerical example below, this restriction may be very important in practical applications.

**3.4. Noisy data of the 1st type.** This section demonstrate the performance of the method for the approximation of noisy data of a function with jump discontinuity. Furthermore, we use this example to emphasize the importance of using the restricted sets in  $\tilde{F}_B$  rather than using  $F_B$ . The underlying function and its noisy version are displayed in Figure 9. In the numerical test we have used the same mesh and knot sizes as in the previous example. In figure 10 we show the results with and without restricting the the set of points which participate in the computation of  $F_B$ . In the left graph we note that the approximation in the inner region is infected by wrong values from the outer region, and this is clearly corrected in the right graph where the least-squares approximations use values which are not to close the discontinuity curve. In Figure 11 we see two approximations to the exact singularity curves (in red), using different knots' grid sizes,  $\delta = 1.5$  and  $\delta = 2.$ , together with the singularity curve of the underlying function. As expected, a smaller  $\delta$  enables higher flexibility of  $g_\gamma$  and a better approximation to the exact curve.



FIGURE 10. Two approximations using different point sets in the least squares approximation

**3.5. 3-corner Jump Discontinuity - Problem C.** Combining elements from Problems A and B, we can now approach the more complex problem of a 3-corner discontinuity. Consider the case of a function defined over a domain  $D$ , with a discontinuity across three curves meeting at a 3-corner, subdividing  $D$  into three sub-domains  $D_1$ ,  $D_2$  and  $D_3$ , as in Figure 12. We assume that  $f|_{D_i}$  is smooth on  $D_i$ ,  $i=1,2,3$ . Following the above discussions, the following procedure is suggested:

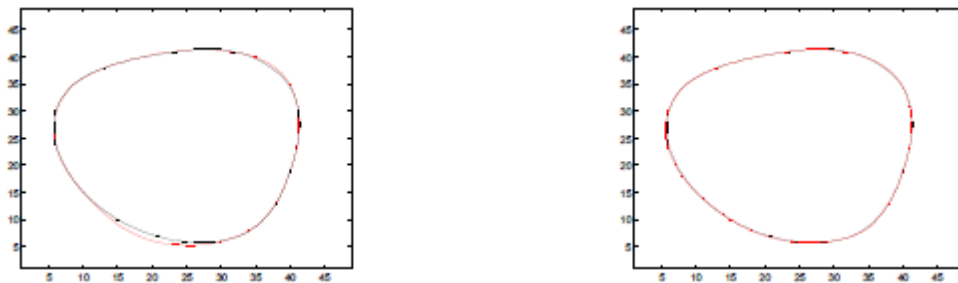


FIGURE 11. Two approximations to the exact singularity curves - using different knots' grid sizes,  $\delta = 2:$  and  $\delta = 1:5$ .

We look for three spline functions,  $\{g_i\}_{i=1}^3$ , approximating  $f$  on  $\{D_i\}_{i=1}^3$  respectively. Here, the approximation of the segmentation into three domains cannot be done via a zero level set approach. Instead, we look for an additional triplet of spline functions,  $\{h_i\}_{i=1}^3$  which define approximations  $\{E_i\}_{i=1}^3$  to  $\{D_i\}_{i=1}^3$  as follows:

$$E_1 = \{x \mid h_1(x) > \max(h_2(x), h_3(x))\},$$

$$E_2 = \{x \mid h_2(x) > \max(h_1(x), h_3(x))\},$$

$$E_3 = \{x \mid h_3(x) > \max(h_1(x), h_2(x))\}.$$

Denoting  $\|u\|_{2,E} = \sum_{x_i \in E} u^2(x_i)$  we would like to minimize the following objective function:

$$(3.8) \quad F_C(p) = \|f - g_1\|_{2,E_1} + \|f - g_2\|_{2,E_2} + \|f - g_3\|_{2,E_3} \rightarrow \text{minimum}.$$

Hence, the segmentation is defined by a *max* operation as in Problem A. Given a segmentation of  $D$  into  $\{E_i\}_{i=1}^3$ , the triplet  $\{g_i\}_{i=1}^3$  is defined, as in Problem B, by a system of linear equations which defines the least-squares solution of (3.8). To achieve better approximation on  $\{D_i\}_{i=1}^3$ , in view of Theorem 3.2, the least-squares approximation for  $\{g_i\}_{i=1}^3$  should exclude data points which are near joint boundaries of  $\{E_i\}_{i=1}^3$ .

For a numerical illustration of Problem C and the approximation obtained by minimization of  $F_C$  we took noisy data from a function with 3-corner discontinuity in  $D = [-2, 2]^2$ . All the unknown spline functions,  $\{g_i\}_{i=1}^3$  and  $\{h_i\}_{i=1}^3$  are bi-cubic with a square grid of knots of grid size  $\delta = 2$ . Since only the splines  $\{h_i\}_{i=1}^3$  enter in a non-linear way into  $F_C$ ,



FIGURE 12. 3-corner discontinuity - noisy data and approximation

the minimization problem involves  $3 \times 9 = 27$  unknowns. As in all the previous examples we have used a differential evolution algorithm for finding an approximate solution of this minimization problem. The noisy data and the resulting approximation are shown in Figure 12.

#### 4. SUMMARY AND ISSUES FOR FURTHER RESEARCH:

We have introduced a unified framework for approximating functions with normals' discontinuity or jump discontinuity across curves. The method may be viewed as an extension of the well known procedures of boolean operations in solid geometry. In this work it is suggested to use a kind of boolean operations on splines as an approximation tool. Through a series of non-trivial examples we have presented the potential of this approach to achieve high quality approximations in many applications. It is interesting to note that all the non-linearity in the suggested approximations can be expressed by the *sign* operation,

or equivalently the  $(\cdot)_+$  operation. The approximation procedure requires high dimensional non-linear optimization, and thus the complexity of computing the approximations is very high. For all the numerical examples in this paper we have used a very powerful matlab code written by Markus Buehren, based upon the method of Differential Evolution ([7]). The execution time ranges from 1 second for the simple univariate problem to 80 seconds for the bivariate Problem C. The differential evolution algorithm usually finds a local minima and not 'the global minimizer'. Yet, as demonstrated, it finds for us very good approximations, and it seems to be robust to noise. A main issue for further study would be the acceleration of the optimization process, e.g., by generating good initial candidates for the optimization. Yet, in spite of the high computational cost, the method may still be very useful for high quality up-sampling, and for functions (or surfaces) with few singularity curves. In a scene with many discontinuities we would suggest to subdivide the domain into patches, each containing one or two singularity curves. Choosing partially overlapping patches, the local approximations can be blended into a global approximation, as demonstrated in Section 3.2. Another simple idea is based upon Corollary 3.3, which tells us to ignore few data points near the approximated singularity curve, in order to attain higher approximation order.

Other important issues for further research would be the following:

- (1) Improved optimization: Here we believe that geometric considerations may be used to significantly accelerate the minimization procedure. Gradient-descent algorithms, similar to those used in [4], may also be helpful here.
- (2) Simple rules for choosing grid size for the splines.
- (3) Other basis functions instead of splines.
- (4) Using  $\ell_1$ -norm instead of the  $\ell_2$ -norm in the objective functions.
- (5) Application to 3D surface data via the moving least-squares method.
- (6) Further approximation analysis.

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