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This Volume of
INSPIRE
is being dedicated to
Aryabhatt: Master Astronomer and Mathematician

Aryabhatt was born in 476 CE (Common Era) in Kusumpur (Bihar). Aryabhatt's intellectual brilliance remapped the boundaries of mathematics and astronomy. In 499 CE, at the age of 23, he wrote a text on astronomy and an unparallel treatise on mathematics called "Aryabhatiyam." Aryabhatt formulated the process of calculating the motion of planets and the time of eclipses. Aryabhatt was the first to proclaim that the earth is round, it rotates on its axis, orbits the sun and is suspended in space – 1000 years before Copernicus published his heliocentric theory. He is also acknowledged for calculating p (Pi) to four decimal places: 3.1416 and the sine table in trigonometry. Centuries later, in 825 CE, the Arab mathematician, Mohammed Ibna Musa credited the value of Pi to the Indians, "This value has been given by the Hindus." And above all, his most spectacular contribution was the concept of zero without which modern computer technology would have been non-existent. Aryabhatt was a colossus in the field of mathematics.

FOREWORD

The present volume of ***INSPIRE*** contains the various research papers of Faculty and Research Scholars of Department of Mathematics, INSTITUTE FOR EXCELLENCE IN HIGHER EDUCATION, BHOPAL (M. P.).

For me it is the realization of a dream which some of us have been nurturing for long and has now taken a concrete shape through the frantic efforts and good wishes of our dedicated band of research workers in our country, in the important area of mathematics.

The editor deserves to be congratulated for this very successful venture. The subject matter has been nicely and systematically presented and is expected to be of use to the workers.

(Dr. Meera Pingle)
Director & Patron
IEHE, Bhopal (M. P.)

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SOME FIXED POINT THEOREMS IN L-SPACES

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ABSTRACT : In the present paper we have given some fixed point theorems in L-spaces . The results obtained generalize many previous results.

KEYWORDS: L-Space, fixed Point, Contraction Mappings, Separated space etc.

MATHEMATICAL SUBJECT CLASIIFICATION : Primary47H10, Secondary 54H25.

1. INTRODUCTION :

Kashara [5] has introduced L- space, then Yeh [8] has give some fixed point theorems in L-Spaces. It seems that the notion of metric is not necessary in the Banach contraction principle and some of its generalizations. The purpose of this paper is to obtained some fixed point results in L-Spaces.

To establish our main result we need the following definitions:

2. PRELIMINARIES :

Definition: 2.1 Let N be the set of all non-negative integers and M be a nonempty set. Then L-Space is defined to be the pair (M, $\rightarrow$) of the set M and a subset $\rightarrow$ of the set $M^N \times M$ satisfying the following two conditions :

(L_1) If $x_n = x \in M$ for all $n \in N$, then $\left(\{x_n\}_{n \in N,}\, x\right) \in \rightarrow$

(L_2) If $\left(\{x_n\}_{n \in N,}\, x\right)$, then $\left(\{x_{n_i}\}_{i \in N,}\, x\right) \in \rightarrow$ for every subsequence $\{x_{n_i}\}_{i \in N,}$ of $\{x_n\}_{n \in N}$.

In what follows instead of writing $\left(\{x_n\}_{n \in N,}\, x\right) \in \rightarrow$, we shall write $\{x_n\}_{n \in N} \rightarrow x$ or $x_n \rightarrow x$ and read $\{x_n\}_{n \in N}$ converges to x. Further we give some definitions regarding L-Space.

Definition: 2.2 A L – space (M, $\rightarrow$) is said to be separated, if each sequence in M converges to at most one point in M .

Definition: 2.3 A mapping f on L – space (M, $\rightarrow$) into an L –space ($M' \rightarrow'$) is said to be continuous if $x_n = x$ implies $f\left(x_{n_i}\right) \rightarrow' f(x)$ for some subsequence $\{x_{n_i}\}_{i \in N,}$ for $\{x_n\}_{n \in N}$.

Definition: 2.4 Let d be a nonnegative extended real valued function on $M \times M$: $0 \le d(x,y) \le \infty$ for all $x, y \in M$. Then the L –space (M, $\rightarrow$) is said to be d-complete if each sequence $\{x_n\}_{n \in N}$ in M with $\sum_{i=0}^{\infty} d(x_i, x_{i+1}) < \infty$ converges to at most one point of M.

3. MAIN RESULTS

Theorem 3.1 : Let $(M, \to)$ be a separated L –space which is d-complete for a non-negative extended real valued function d on $M \times M$ with $d(x, x) = 0$ for each x in M and f be a continuous self mapping satisfying the following two conditions :

(3.1.1) There exist a_i (I = 1, 2,... ,5), p and q with $\sum_{i=1}^{5} a_i > p$, $p - a_2 \geq 0$, $0 < q \leq \infty$ such that

$$a_1 d(fx, fy) + a_2 d(x, fx) + a_3 d(y, fy) + a_4 d(fx, f^2x) + a_5 d(y, f^2x) - \min\{d(x, fy), d(y, fx), d(x, f^2x), d(fy, f^2x)\} \leq pd(x, y),$$

for x, y in M with $d(x, y) < q$;

(3.1.2) There exist $u \in M$ such that $d(u, fu) < q$.

Then f has a fixed point and the sequence $\{f^n u\}_{n\in N}$ converges to the fixed point .

PROOF : Let $x_0 = u$, $x_n = f x_{n-1}$ for n = 1, 2, ... It follows from (3.1.1) and (3.1.2) that

$$a_1 d(x_n, x_{n+1}) + a_2 d(x_{n-1}, x_n) + a_3 d(x_n, x_{n+1}) + a_4 d(x_n, x_{n+1}) + a_5 d(x_n, x_{n+1}) - \min\{d(x_{n-1}, x_{n+1}), d(x_n, x_n), d(x_{n-1}, x_{n+1}), d(x_{n+1}, x_{n+1})\} \leq pd(x_{n-1}, x_n)$$

Or, $$d(x_n, x_{n+1}) \leq \frac{p - a_2}{a_1 + a_3 + a_4 + a_5} d(x_{n-1}, x_n) \quad \text{. . . (3.1.3)}$$

Then by induction, we have

$$d(x_n, x_{n+1}) \leq \left(\frac{p - a_2}{a_1 + a_3 + a_4 + a_5}\right)^n d(u, fu) \quad \text{. . . (3.1.4)}$$

for every n in N. Therefore we have $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty$.

Thus the d-completeness of M implies that the sequence $\{f^n u\}_{n\in N}$ converges to some $z \in M$. Hence by the continuity of f, there is a subsequence $\{f^{n_i} u\}_{i\in N}$ of $\{f^n u\}_{n\in N}$, such that

$f(f^{n_i} u) \to fz$.

But, since that $\{f(f^{n_i} u)\}_{i\in N}$ is a subsequence of $\{f^n u\}_{n\in N}$, hence we have $f(f^{n_i} u) \to z$.

So that $fz = z$. This completes the proof of the theorem.

Theorem 3.2 : Let $(M, \to)$ be an L –space which is d-complete for a continuous non-negative extended real valued function d on $M \times M$ with the properties :

(3.2.1) $d(x, y) = 0$ implies that $x = y$;

(3.2.2) $d(x, x) = 0$ for each $x \in M$

If f is continuous self mapping of M satisfying (3.1.1) and (3.1.2), then f has a fixed point .

PROOF : As in the proof of theorem 3.1, we have by (3.1.4)

$$d(x_n, x_{n+1}) \leq \left(\frac{p - a_2}{a_1 + a_3 + a_4 + a_5}\right)^n d(u, fu) \quad \text{........ (3.1.5)}$$

holds for every n in N and the sequence $\{f^n u\}_{n\in N}$ converges to some z ∈ M and that

$$f\,(f^{n_i}u) \to fz$$

for some subsequence $\{f^{n_i}u\}_{i\in N}$ of $\{f^n u\}_{n\in N}$.
Therefore by the continuity of f we have

$$d(f\,(f^{n_i}u),\ (f^{n_i}u)\,) \to d(fz, z)$$

for some subsequence $\{f^{n_i}u\}_{i\in N}$ of $\{f^n u\}_{n\in N}$.
Thus, $d(f\,(f^{n_i}u),\ (f^{n_i}u)\,) \to 0$.
So that d(fz, z) = 0 and therefore, fz = z, i.e. z be a fixed point of f.

REFERENCES

[1] Bhardwaj, R., Agarwal, S.,and Yadav, R. N., Some results on L –spaces, International Journal of Theoritical & Applied Science, 1(1), 120 – 127 (2009).
[2] Gupta, Animesh, Jaiswal, Dilip and Rajput, S. S.,Some fixed point and common fixed point results in L –space with rational contraction, International Journal of Theoritical & Applied Science, 3(2), 52 – 56 (2011).
[3] Jaggi, D.S., Some unique fixed point theorems, Indian J.Pure and Applied Math.,8 (2), 223-230 (1977).
[4] Jaroslaw Gornicki, Fixed point theorems for Kannan type mappings, Journal of Fixed Point Theory and Applications, 19, 2145 – 2152 (2017).
[5] Kasahara, S., On some generalizations of the Banach contraction theorem, RIMS,Kyoto Univ., 12, 427-437(1976).
[6] Rhoades, B.E., Generalized contractions, Bull.Calcutta Math. Soc., 71, 323-330 (1979).
[7] Saluja, A.S. and Dhakde, Alkesh Kumar, Some fixed point theorems on L – spaces, International Journal of Mathematical Archive, 5 (2), 1 – 8 (2014).
[8] Yeh,C. C.,Some fixed point theorems in L –spaces, Indian J.Pure and Applied Math.,9(10), 993-995 (1978).

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Fixed Point Theorem on Partial Metric Spaces

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1. Introduction and preliminaries

Let (X,d) be a metric space and $CB(X)$ denotes the collection of all nonempty closed and bounded subsets of X. For$A$, $B\in CB(X)$, define

$H(A,B)$ =max$\{\sup_{a\epsilon A} d(a,B), \sup_{b\epsilon B} d(b,A),\}$, where $d(x, A) = \inf\{d(x,a): a \in A\}$ is the distance of a point x to the set A. It is known that H is a metric on $CB(X)$, called the Hausdorff metric induced by the metric d.

Definition 1.1.[5] Let X be any nonempty set. An element x in X is said to be a fixed point of a multi-valued mapping $T : X \to 2^X$,if $x \in T x$, where 2^X denotes the collection of all nonempty subsets of X.

A multi-valued mapping $T : X \to CB(X)$ is said to be a contraction if , $H(Tx, Ty)\leq kd(x, y)$, for all $x, y \in X$ and for some k in $[0,1)$.

The study of fixed points for multi-valued contractions using the Hausdorff metric was initiated by Nadler [15] who proved the following theorem.

Theorem 1.2. *([15]) Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a contraction mapping. Then, there exists $x \in X$ such that $x \in Tx$.*

Later, an interesting and rich fixed point theory was developed. The theory of multi-valued maps has application in control theory, convex optimization, differential equations and economics (see also [7]). On the other hand, Matthews [10] introduced the concept of a partial metric as a part of the study of denotational semantics of dataflow networks. He gave a modified version of the Banach contraction principle, more suitable in this context (see also [8,11]). In fact, (complete) partial metric spaces constitute a suitable framework to model several distinguished examples of the theory of computation and also to model metric spaces via domain theory (see, [6,9,10,11–14]).

Consistent with [2,3,10], the following definitions and results will be needed in the sequel.

Definition 1.3. [5] Let X be a nonempty set. A function $p : X \times X \to R+$ is said to be a partial metric on X if for any $x, y, z \in X$, the following conditions hold:

(P_1) $p(x, x) = p(y, y) = p(x, y)$, if and only if $x = y$;

(P_2) $p(x, x) \leq p(x, y)$

(P_3) $p(x,y) = p(y, x)$

(P_4) $p(x, y) \leq p(x, y) + p(y, z) - p(y, y)$

The pair (X, p) is then called a partial metric space.

If $p(x, y) = 0$, then (P_1) and (P_2) imply that $x = y$. But the converse does not hold always.

A trivial example of a partial metric space is the pair (R+, p), where $p : R^+ \times R^+ \to R+$ is defined as $p(x, y) = \max\{ x, y \}$(see also [1]).

Example 1.4. ([10]) If $X = \{[a,b] : a,b \in R, a\ b \}$, then $p([a,b] , [c,d]) = \max\{b,d\} - \min\{a,c\}$ defines a partial metric p on X.

For some more examples of partial metric spaces, we refer to [2,4,12,14].

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base of the family open p-balls $\{ B_p(x,\varepsilon): x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{ y \in X: p(x, y) < p(x, x) + \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.

Observe (see [10, p. 187]) that a sequence $\{ x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$, with respect to τ_p, if and only if p is a partial metric on X, $p(x' x)= \lim_{n\to\infty} p(x' x_n)$, then the function $p^s: X \times X \to R^+$ given by

$_p{}^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, defines a metric on X.

Furthermore, a sequence $\{ x_n\}$ converges in (X, p^s) to a point $x \in X$ if and only if

$$\lim_{n,m\to\infty} p(x_n,x_m) = \lim_{n\to\infty} p(x_n,x) = p(x,x). \tag{1.1}$$

Definition 1.5. ([10]) Let (X, p) be a partial metric space.

(a) A sequence $\{ x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.

(b) (X, p) is said to be complete if every Cauchy sequence $\{ x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $\lim_{n\to\infty} p(x, x_n) = p(x, x)$. In this case, we say that the partial metric p is complete.

Lemma 1.6. ([2,10]) Let (X, p) be a partial metric space. Then:

(a) A sequence $\{ x_n\}$ in X is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in metric space (X, p^s).

(b) A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete.

2. Partial Hausdorff metric

Let (X, p) be a partial metric space. Let $CB^p(X)$ be the family of all non-empty, closed and bounded subsets of the partial metric space (X, p), induced by the partial metric p. Note that Closedness is taken from (X,τ_p) (τ_p is the topology induced by p) and boundedness is given as follows:

A is a bounded subset in (X, p) if there exist $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0, M)$, that is, $p(x_0,a) < p(a,a) + M$.

For $A, B \in CB^p(X)$ and $x \in X$, define

$$p(x, A) = \inf\{p(x, a),\ a \in A\}, \qquad \delta_p(A, B) = \sup\{p(a, B) : a \in A\} \quad \text{and}$$

$$\delta_p(B, A) = \sup\{p(b, A) : b \in B\}.$$

It is immediate to check that $p(x, A) = 0 \Rightarrow p^s(x, A) = 0$ where $p^s(x, A) = \inf\{ p^s(x,a), a \in A \}$._

Remark 2.1. ([2]) Let (X, p) be a partial metric space and A any nonempty set in (X, p), then $a \in \overline{A}$ if and only if $p(a, A) = p(a,a)$, (2.1)

Where, A denotes the closure of A with respect to the partial metric p. Note that A is closed in (X, p) if and only if $A = \overline{A}$

Now, we shall study some properties of mapping $\delta_p: CB^p(X) \times CB^p(X) \rightarrow [0, \infty)$.

Proposition 2.2.[5] *Let (X, p) be a partial metric space. For any A, B, C ∈ $CB^p(X)$, we have the following*:

(i) (i) $\delta p(A, A) = \sup\{p(a, a) : a \in A\}$;

(ii) (ii) $\delta p(A, A) \leq \delta p(A, B)$;

(iii) (iii) $\delta p(A, B) = 0$ *implies that* $A \subseteq B$;

(iv) (iv) $\delta p(A, B) \leq \delta p(A, C) + \delta p(C, B) - \inf_{c \in C} p(c, c)$.

Proposition 2.3.[5] *Let (X, p) be a partial metric space. For all A, B, C ∈ $CB^p(X)$, we have*

(h1) $H_p(A, A) \leqslant H_p(A, B)$;

(h2) $H_p(A, B) = H_p(B, A)$;

(h3) $H_p(A, B) \leqslant H_p(A, C) + H_p(C, B) - \inf_{c \in C} p(c,c)$.

Corollary 2.4. [5] *Let (X, p) be a partial metric space. For A, B ∈ $CB^p(X)$ the following holds*

$H_p(A, B) = 0$ *implies that* $A = B$.

Remark 2.5.[5] The converse of Corollary 2.4 is not true in general as it is clear from the following example.

Example 2.6.[5] Let $X = [0,1]$ be endowed with the partial metric $p : X \times X \rightarrow R+$ defined by $p(x, y) = \max\{ x, y \}$.

From (i) of Proposition 2.2, we have

$$H_p(X, X) = \delta_p(X, X) = \sup\{ x: 0\ x\ 1\} = 1 \neq 0.$$

In view of Proposition 2.3 and Corollary 2.4, we call the mapping H_p: $CB^p(X) \times CB^p(X) \to [0, +\infty)$, a partial Hausdorff metric induced by p.

Remark 2.7.[5] It is easy to show that any Hausdorff metric is a partial Hausdorff metric. The converse is not true (see Example 2.6).

3. Fixed point of multi-valued contraction mapping

We start with the following lemma needed to prove our main result.

Lemma 3.1.[5] *Let* (X, p) *be a partial metric space,* $A, B \in CB^p(X)$ *and* $h > 1$. *For any* $a \in A$, *there exists* $b = b(a) \in B$ *such that*

$$p(a,b) \leq hH_p(A, B). \tag{3.1}$$

Aydi et al[5] proved the following:

Theorem 3.2. *Let* (X, p) *be a complete partial metric space. If* $T : X \to CB^p(X)$ *is a multi-valued mapping such that for all* $x, y \in X$, *we have*

$$H_p(Tx, Ty) \leqslant kp(x, y) \tag{3.2}$$

where $k \in (0,1)$. *Then T has a fixed point.*

Now let us prove our main result,

Theorem 3.3. *Let* (X, p) *be a complete partial metric space. If* $T : X \to CB^p(X)$ *is a multi-valued mapping such that for all* $x, y \in X$, *we have*

$$H_p(Tx, Ty) \leq k \max\{p(x, y), p(x, Tx), p(y, Tx)\}$$

where $k \in (0,1)$. *Then T has a fixed point.*

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$. From Lemma 3.1 with $h = \frac{1}{\sqrt{k}}$, there exists $x_2 \in Tx_1$ such that $p(x_1, x_2) \leq \frac{1}{\sqrt{k}} H_p(Tx_0, T x_1)$.

$$H_p(Tx_0, T x_1) \leq k \max\{p(x_0, x_1), p\big(x_0, Tx_0, p(x_1, Tx_0)\big)\}$$

$$\frac{1}{\sqrt{k}} H_p(Tx_0, T x_1) \leq \frac{k}{\sqrt{k}} p(x_0, x_1) = \sqrt{k} p(x_0, x_1)$$

$p(x_1, x_2) \leq \sqrt{k} p(x_0, x_1)$. For $x_2 \in Tx_1$, there exists $x_3 \in Tx_2$ such that $p(x_2, x_3) \leq \sqrt{k}(Tx_1, T x_2)$. *Continuing this process, we obtain a sequence* $\{x_n\}$ *in X such that*

$$x_{n+1} \in Tx_n \text{ and } p(x_{n+1}, x_n) \leq \sqrt{k} p(x_n, x_{n-1}) \text{ for all } n \geq 1. \tag{3.3}$$

Now from (3.3) and by mathematical induction, we obtain

$$p(x_{n+1}, x_n) \leq (\sqrt{k}^{\,n}) p(x_0, x_1) \text{ for all } n \in \mathbb{N}. \tag{3.4}$$

Using (3.4) and the property (P_4) of a partial metric, for any $m \in \mathbb{N}^*$, we have

$$\begin{aligned} p(x_n, x_{n+m}) &\leqslant p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{n+m-1}, x_{n+m}) \\ &(\sqrt{k})^n p(x_0, x_1) + (\sqrt{k})^{n+1} p(x_0, x_1) + \cdots + (\sqrt{k})^{n+m-1} p(x_0, x_1) \\ &\big((\sqrt{k})^n \quad (\sqrt{k})^{n+1} \quad\quad (\sqrt{k})^{n+m-1}\big) p(x_0, x_1) \\ &\frac{}{1 - \sqrt{k}} p(x_0, x_1) \to 0 \quad \text{as } n \to +\infty \text{ (since } 0 < k < 1). \end{aligned}$$

By the definition of p^s, we get for any $m \in \mathbb{N}^*$,

$$p^s(x_n, x_{n+m}) \leqslant 2p(x_n, x_{n+m}) \to 0 \quad \text{as } n \to +\infty. \qquad (3.5)$$

This yields that $\{x_n\}$ is a Cauchy sequence in (X, p^s). Since (X, p) is complete, then from Lemma 1.6, (X, p^s) is a complete metric space. Therefore, the sequence $\{x_n\}$ converges to some $x^* \in X$ with respect to the metric p^s, that is, $\lim_{n\to+\infty} p^s(x_n, x^*) = 0$. Again, from (1.1), we have

$$p(x^*, x^*) = \lim_{n\to+\infty} p(x_n, x^*) = \lim_{n\to+\infty} p(x_n, x_n) = 0. \qquad (3.6)$$

Since $H_p(Tx_n, Tx^*) \leqslant kp(x_n, x^*)$, therefore

$$\lim_{n\to+\infty} H_p(Tx_n, Tx^*) = 0 \qquad (3.7)$$

Now $x_{n+1} \in Tx_n$ gives that

$$p(x_{n+1}, Tx^*) \leqslant \delta_p(Tx_n, Tx^*) H_p(Tx_n, Tx^*).$$

From (3.7), we get

$$\lim_{n\to+\infty} p(x_{n+1}, Tx^*) = 0 \qquad (3.8)$$

On the other hand, we have

$$p(x^*, Tx^*) \leqslant p(x^*, x_{n+1}) + p(x_{n+1}, Tx^*).$$

Taking limit as $n \to +\infty$ and using (3.6) and (3.8), we obtain $p(x^*, Tx^*) = 0$.

Therefore, from (3.6) ($p(x^*, x^*) = 0$), we obtain

$$p(x^*, x^*) = p(x^*, Tx^*),$$

which from (2.1) implies that $x^* \in \overline{Tx^*} = Tx^*$. □

To underline the usefulness of partial metric, we give the following very simple illustrative examples.

Example 3.3. [5] Let $X = \{0,1,4\}$ be endowed with the partial metric $p : X \times X \to \mathbb{R}^+$ defined by

$$p(x, y) = \frac{1}{4}|x - y| + \frac{1}{2}\max\{x, y\} \text{ for all } x, y \in X.$$

Note that $p(1,1) = \frac{1}{2} \neq 0$ and $p(4,4) = 2 = 0$, so p is not a metric on X.

As $p^s(x, y) = |x - y|$ so (X, p) is a complete partial metric space.

Note that $\{0\}$ and $\{0,1\}$ are bounded sets in (X, p). Infact, if $x \in \{0,1, 4\}$, then $x \in \overline{\{0\}} \Leftrightarrow p(x, \{0\}) = p(x, x) \Leftrightarrow \frac{3}{4}x = \frac{1}{2}x$

$\Leftrightarrow x = 0$

$\Leftrightarrow x \in \{0\}$.

Hence $\{0\}$ is closed with respect to the partial metric p. Also

$x \in \overline{\{0, 1\}} \Leftrightarrow p(x, \{0, 1\}) = p(x, x)$

$$\Leftrightarrow \min\left\{\frac{3}{4}x, \frac{1}{4}|x - 1| + \frac{1}{2}\max\{x, 1\}\right\} = \frac{1}{2}x$$

$\Leftrightarrow$ $x \in \{0,1\}$.

Hence {0,1} is closed with respect to the partial metric T: $X \to CBp(X)$ by p. Now, define the mapping

$T(0) = T(1) = \{0\}$ and $T(4) = \{0,1\}$.

We shall show that, for all $x, y \in X$, the contractive condition (3.2) is satisfied with $k = \frac{1}{2}$. For this, we consider the following cases:

- $x, y \in \{0,1\}$. We have $H_p\big(T(x), T(y)\big) = H_p\big(\{0\}, \{0\}\big) = 0$, and (3.2) is satisfied obviously.
- $x \in \{0,1\}$, $y = 4$. We have

$$\begin{aligned} H_p\big(T(0), T(4)\big) &= H_p\big(T(1), T(4)\big) \\ &= H_p\big(\{0\}, \{0, 1\}\big) \\ &= \max\big\{p\big(0, \{0, 1\}\big), \max\big\{p(0, 0), p(1, 0)\big\}\big\} \\ &= \frac{3}{4} \leqslant \frac{11}{8} = kp(1, 4) < \frac{3}{2} = kp(0, 4). \end{aligned}$$

- $x = y = 4$. We have

$$\begin{aligned} H_p\big(T(4), T(4)\big) &= H_p\big(\{0, 1\}, \{0, 1\}\big) \\ &= \sup\big\{p(x, x)\colon x \in \{0, 1\}\big\} \\ &= \max\big\{p(0, 0), p(1, 1)\big\} \\ &= \frac{1}{2} \leqslant 1 = kp(4, 4). \end{aligned}$$

Thus, all the hypotheses of Theorem 3.2 are satisfied. Here, $x = 0$ is a fixed point of T.

Example 3.4.[5] Let $X = \{0,1,2\}$ be endowed with the partial metric $p : X \times X \to R+$ defined by

$$p(0,0) = p(1,1) = 0,\ p(2, 2) = \frac{1}{4},$$

p(0, 1) = p(1, 0) = 1/3

p(0, 2) = p(2, 0) = 11/24

p(1, 2) = p(2, 1) = 1/2

Define the mapping $T : X \to CB^p(X)$ by

$T(0) = T(1) = \{0\}$ and $T(2) = \{0,1\}$.

Note that, $T x$ is closed and bounded for all $x \in X$ under the given partial metric space (X, p). We shall show that, for all $x, y \in X$, the contractive condition (3.2) is satisfied with $k = \frac{3}{4}$. For this, we consider the following cases:

- $x, y \in \{0,1\}$. We have
  $$H_p(T(x), T(y)) = H_p(\{0\}, \{0\}) = 0,$$
  and (3.2) is satisfied obviously.
- $x \in \{0,1\}$, $y = 2$. We have
  $$\begin{aligned} H_p\ (T(1), T(2)) &= H_p(T(0), T(2)) \\ &= H_p(\{0\}, \{0, 1\}) \\ &= \max\{p(0, \{0, 1\}), \max\{p((\\ &= \frac{1}{3} \leqslant \frac{11}{24}k = kp(0, 2) < \frac{1}{2}k \end{aligned}$$
- $x = y = 2$. We have
- $$\begin{aligned} H_p\ (T(2), T(2)) &= H_p(\{0, 1\}, \{0, 1\}) \\ &= \sup\{p(x, x) \colon x \in \{0, 1\}\} \\ &= \max\{p(0, 0), p(1, 1)\} \\ &= 0 < \frac{1}{4}k = kp(2, 2). \end{aligned}$$

Thus, all the conditions of Theorem 3.2 are satisfied. Here, $x = 0$ is a fixed point of T . On the other hand, the metric p^sinduced by the partial metric p is given by $p^s(0,0) = p^s(1,1) = p^s(2,2) = 0$,

$$p^s\ (0, 1) = p^s(1, 0) = p^s(0, 2) = p^s(2, 0) = \frac{2}{3}$$

$$p^s\ (2, 1) = p^s(1, 2) = \frac{3}{4}$$

Now, it is easy to show that Theorem 1.2 is not applicable in this case. Indeed, for $x = 0$ and $y = 2$, we have

$$\begin{aligned} H(T(0), T(2)) &= H(\{0\}, \{0, 1\}) \\ &= \max\{\sup\{p^s(0, \{0, 1\})\}, \sup\{p^s(\{0, 1\}, \{0\})\}\} \\ &= \max\left\{0, \frac{2}{3}\right\} = \frac{2}{3} \nleqslant \frac{2}{3}k = kp^s(0, 2), \end{aligned}$$

for any $k \in (0,1)$.

REFERENCES

[1] M. Abbas, T. Nazir, Fixed point of generalized weakly contractive mappings in ordered partial metric spaces, Fixed Point Theory Appl. 2012 (1) (2012) 19 pp.

[2] I. Altun, H. Simsek, Some fixed point theorems on dualistic partial metric spaces, J. Adv. Math. Stud. 1 (2008) 1–8.

[3] I. Altun, F. Sola, H. Simsek, Generalized contractions on partial metric spaces, Topology Appl. 157 (2010) 2778–2785.

[4] M.A. Bukatin, S.Yu. Shorina, Partial metrics and co-continuous valuations, in: M. Nivat, et al. (Eds.), Foundations of Software Science and Computation Structure, in: Lecture Notes in Comput. Sci., vol. 1378, Springer, 1998, pp. 125–139.

[5] H. Aydi, M. Abbas, C. Vetro, Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces,159(2012), 3234-3242.

[6] L.j. Ciri´ c, B. Samet, H. Aydi, C. Vetro, Common fixed points of generalized contractions on partial metric spaces and an application, Appl. Math.´ Comput. 218 (2011) 2398–2406.

[7] B. Damjanovic, B. Samet, C. Vetro, Common fixed point theorems for multi-valued maps, Acta Math. Sci. Ser. B Engl. Ed. 32 (2012) 818–824.

[8] C. Di Bari, P. Vetro, Fixed points for weak φ-contractions on partial metric spaces, Int. J. Eng., Contemp. Math. Sci. 1 (1) (2011) 5–13.

[9] R. Heckmann, Approximation of metric spaces by partial metric spaces, Appl. Categ. Structures 7 (1999) 71–83.

[10] S.G. Matthews, Partial metric topology, in: Proc. 8th Summer Conference on General Topology and Applications, in: Ann. New York Acad. Sci., vol. 728, 1994, pp. 183–197.

[11] D. Paesano, P. Vetro, Suzuki's type characterizations of completeness for partial metric spaces and fixed points for partially ordered metric spaces, Topology Appl. 159 (3) (2012) 911–920.

[12] S. Romaguera, A Kirk type characterization of completeness for partial metric spaces, Fixed Point Theory Appl. 2010 (2010), Article ID 493298, 6 pp.

[13] S. Romaguera, O. Valero, A quantitative computational model for complete partial metric spaces via formal balls, Math. Structures Comput. Sci. 19 (2009) 541–563.

[14] M.P. Schellekens, The correspondence between partial metrics and semivaluations, Theoret. Comput. Sci. 315 (2004) 135–149.

[15] S.B. Nadler, Multivalued contraction mappings, Pacific J. Math. 30 (1969) 475–488.

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ENCRYPTION AND DECRYPTION TECHNIQUE INVOLVING METRIC SPACE

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ABSTRACT

The aim of this paper is to establish a technique for encryption and decryption depends on the Connected Components of a Metric Space. We consider a basis for a Metric Space such that the elements of this basis are the Components of that Metric Space.

Key Words: Metric Space, Connected Components, Basis, Encryption, Decryption.

1. INTRODUCTION:

The study of secure communications techniques is called Cryptography that permits to view a message and its contents only to the sender and recipient. Cryptography is closely related to encryption.

The original message is known as plaintext, and ciphertext is the decrypted message. Plaintext and the ciphertext both are written in the terms of elements from a finite set A, called an alphabet of description.

2. RELATED WORK:

Certain encryption and decryption techniques of a message involving group theory, metric space and topological space have been established by Rani [1], Okelo [2], Mahdi [3], Sharma [4], Arora [5], Kahrobaei [6], Iswariya [7] and others.

Looking importance and usefulness of encryption and decryption techniques of a message, we propose to establish a new encryption and decryption techniques of a message involving Metric Space following on the lines of above authors.

3. BASIC CONCEPTS:

Metric Space:

Let X be a non-empty set. A mapping $d : X \times X \to R$ which maps $X \times X$ into R (the set of reals) is said to be a metric (or distance function) if d satisfies the following axioms:

1. $d(x, y) = 0 \Leftrightarrow x = y \quad \forall\, x, y \in X$
2. $d(x, y) = d(y, x) \quad \forall\, x, y \in X$
3. $d(x, y) \leq d(x, z) + d(z, y) \quad \forall\, x, y, z \in X$

If d is a metric for X, then the ordered pair (X, d) is called a Metric Space and d(x, y) is called the distance between x and y.

Separated Sets:

Let (X, d) be a metric space and A, B $\subseteq$ X. The sets A and B are said to be separated if A $\cap$ B = ϕ and $\bar{A} \cap$ B = ϕ.

Disconnected and Connected Sets:

Let (X, d) be a metric space. X is said to be disconnected if it can be expressed as the union of two non-empty separated sets. Furthermore, X is said to be connected if it not disconnected, i.e. X cannot be expressed as the union of two non-empty separated sets.

Let (X, d) be a metric space and Y $\subseteq$ X.

(i) The subset Y of X is said to be disconnected if it is disconnected as a subspace (X, d).

(ii) A subspace (subset) is said to be connected if I is not disconnected.

Components:

Let (X, d) be a metric space and x $\in$ X. The largest connected subset of X containing x, denoted by C(x) is called a component of X containing x. In other words, C(x) is the union of all connected subsets of X each of which contains x.

The components of metric space X are connected disjoint subspaces of metric space X whose union is metric space X, such that each non-empty connected subspace of metric space X intersects only one of them.

Example: Let X be the metric space of all rational numbers with the usual metric d(x, y) = |x − y| and let E $\subseteq$ X. If E has more than one element, we can choose distinct x, y $\in$ E. Further, we can choose an irrational number r between x and y. Let A = {z $\in$ E : z < r), B = {z $\in$ E : z > r}. Then {A, B} becomes a disconnection of E. Hence, E is connected if and only if E is a singleton. Thus, E is a component if and only if it is a singleton. We particularly notice that X is disjoint union of its components.

4. METHODOLOGY:

Consider a message M. Let a finite metric space X s.t. the number of elements depends on the number of character of that message which we need to be encode. Let X such that its components form a basis for its metric space. Each component has the same number as the number of letters.

Let a finite set A of different numbers. Then for each letter of the message M, we assign a number of A. Now we send this number to a fixed vector of a set of vectors Y. This fixed vector is sent to a component of X.

Here, we used the following linear transformation

$f : R^n \longrightarrow R^n$,

f (u) = Bu

where B is a nonsingular matrix and columns of B are the components of X.

Encryption and decryption processes

M	A		Y		V		C
m_1	1	$\leftrightarrow$	y_1	$\leftrightarrow$	v(1)	$\leftrightarrow$	c_1
m_2	2	$\leftrightarrow$	y_2	$\leftrightarrow$	v(2)	$\leftrightarrow$	c_2
	:				:		:
m_n	n	$\leftrightarrow$	y_n	$\leftrightarrow$	v(n)	$\leftrightarrow$	c_n

where

(i) Message space M is a set of strings or plain text messages over some alphabet, which needs to be encrypted.

(ii) A is the set of different numbers. Each letter of the message is assigned by a number, where

$$|M| = |A| = |Y| = |V| = |C| = n,\ |X| = n^2.$$

(iii) X is a Metric space with a basis $d_{\mathcal{B}}$.

(iv) Y is transform the message to the vector of secure number.

(v) V is the vector transform to the component of X.

(vi) C denotes a vector which is known as cipher vector.

5. CRYPTOGRAPHIC ANALYSIS:

THE ENECRYPTION PROCESS (ALGORITHM)

$R \xrightarrow{h} R^n \xrightarrow{g} R^n \xrightarrow{f} R^n$

$A \to Y \to V \to C$

$(f \circ g \circ h) : R \to R^n$

$(f \circ g \circ h)$ is one-to-one transformation from A onto C.

$(f \circ g \circ h)(x_i)$

$= f(g(h(x_i)))$

$= f(g(y_i))$ [where $x_i \in A$, $y_i \in Y$, and $h : A \to Y$, therefore $h(x_i) = y_i$]

$= f(v(i))$ [where $y_i \in Y$, $v(i) \in V$, and $g : Y \to V$, therefore $g(y_i) = v(i)$]

$= Bv(i)$

$= c_i$

THE DECRYPTION PROCESS (ALGORITHM)

$R^n \xrightarrow{f} R^n \xrightarrow{g} R^n \xrightarrow{h} R$

$C \to V \to Y \to A$

$(f \circ g \circ h)^{-1} : R^n \to R$

$(f \circ g \circ h)^{-1}$ is one-to-one transformation from C onto A.

$(f \circ g \circ h)^{-1} (c_i)$
$= (h^{-1} o g^{-1} o f^{-1})(c_i)$
$= h^{-1}(g^{-1}(f^{-1}(c_i)))$
[where $x_i \in A$, $y_i \in Y$, and $h : A \rightarrow Y$, therefore $h(x_i) = y_i$]
$= h^{-1}(g^{-1}(B^{-1}(c_i)))$
[where $y_i \in Y$, $v(i) \in V$, and $g : Y \rightarrow V$, therefore $g(y_i) = v(i)$]
$= h^{-1}(g^{-1}(v(i)))$
$= h^{-1}(y_i)$
$= x_i$

6. CONCLUSION:

We used a basis for a topological space and a linear transformation to encrypt a message. This method or cryptography was a new proposed method.

REFERENCES

1. Rani Asha, Jyoti Kumari: A New Approach to Public Key Transferring Algorithm Scheme Using Metric Space, IOSR Journal of Mathematics (IOSR-JM) e-ISSN: 2278-5728, p-ISSN: 2319-765X, Volume 12, Issue 1 Ver. III (Jan. - Feb. 2016), pp. 04-06.
2. Okelo N. B.: Properties of Complete Metric Spaces and their Applications to Computer Security, Information and Communication Technology: Evolving Trends in Engineering and Technology Online: 2014-08-04, ISSN: 2349-915X, Vol. 1, pp 23-28.
3. Mahdi Hamza Abedal-Hamza, Al-khafaji Saba Nazar Faisel: Using the Connected Components of Topological Spaces in Cryptography, International Journal of Mathematics Trends and Technology, Volume 12, Number 1, Aug 2014, pp. 31-33.
4. Sharma P. L., Rehan M.: On Security of Hill Cipher using Finite Fields, International Journal of Computer Applications (0975 – 8887) Volume 71– No.4, May 2013, pp. 30-33.
5. Arora Priya, Use of Group Theory in Cryptography, IJARIIE-ISSN(O)-2395-4396, Vol-2, Issue-6, 2016, pp. 1767-1772.
6. Kahrobaei Delaram, Anshel Michael: Applications of Group Theory in Cryptography, International Journal of Pure and Applied Mathematics, Volume 58, No. 1, 2010, pp. 21-23.
7. Iswariya S., Rishivarman A. R.: An Arithmetic Technique for Non-Abelian Group Cryptosystem, International Journal of Computer Applications (0975 – 8887), Volume 161, No 2, March 2017, pp. 32-35.

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Expansion Formula Involving Generalized Hypergeometric Function

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Abstract

In this paper, we present and solve a two dimensional Exponential Bessel differential equation, and obtain a particular solution of it involving Fox's H-function.

1. Introduction:

The object of this paper is to formulate a two dimensional Exponential-Bessel partial differential equation and obtain its double series solution. We further present a particular solution of our Exponential-Bessel equation involving Fox's H-function. It is interesting to note that particular solution also yields a new two dimensional series expansion for Fox's H-function involving exponential functions and Bessel functions.

H-function of one variable which is introduced by Fox [5, p.408], will be represented as follows:

$$H^{m,n}_{p,q}\left[x \left| \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right.\right] = (1/2\pi i) \int_L \theta(s)\, x^s\, ds \tag{1.1}$$

where $i = \sqrt{(-1)}$,

$$\theta(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j s) \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j s)}$$

x is not equal to zero and an empty product is interpreted as unity; p, q, m, n are integers satisfying $1 \le m \le q$, $0 \le n \le p$, α_j (j = 1, …., p), β_j (j = 1, …, q) are positive numbers and a_j (j = 1, …, q) are complex numbers. L is a suitable contour of Barnes type such that poles of $\Gamma(b_j - \beta_j s)$ (j = 1, …, m) lie to the right and poles of $\Gamma(1 - a_j + \alpha_j s)$ (j = 1, …, n) to the left of L. These assumptions for the H-function will be adhered to through out this paper.

According to Braakasma

$$H^{m,n}_{p,q}\left[x \left| \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right.\right] = O(|x|^{\alpha}) \text{ for small } x,$$

where $\sum_{j=1}^{p} \alpha_j - \sum_{j=1}^{q} \beta_j \le 0$ and $\alpha = \min R(b_h/\beta_h)$ (h = 1, .., k) and

$$H^{m,n}_{p,q}\left[x \,\middle|\, \begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right] = O(|x|^{\beta}) \text{ for large } x,$$

where

$$\sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p}\alpha_j + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q}\beta_j \equiv A > 0,$$

$$\sum_{j=1}^{p}\alpha_j - \sum_{j=1}^{q}\beta_j < 0$$

$|\arg x| < \frac{1}{2} A\pi$ and $\beta = \max R[(a_j - 1)/\alpha_j]$ $(j = 1, .., n)$

The following formulae are required in the proof:

The integral [2, p.704, (2.2)] (modified form):

$$\int_0^{\pi} \cos 2ux\,(\sin x/2)^{-2w} H^{m,n}_{p,q}\left[z(\sin x/2)^{2h} \,\middle|\, \begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right] dx$$

$$= \sqrt{\pi}\, H^{m+1,n+1}_{p+2,q+2}\left[z \,\middle|\, \begin{matrix}(1/2+w,h),(a_j,\alpha_j)_{1,p},(w,h)\\(2u+w,h),(b_j,\beta_j)_{1,q},(w-2u,h)\end{matrix}\right] \tag{1.2}$$

where $h > 0$, $|\arg z| < \frac{1}{2} A\pi$, $\mathrm{Re}(1-2w) - 2h \max\limits_{1\le j\le n}[\mathrm{Re}(a_j - 1)/\alpha_j] > 0$.

The integral [7, p. 94, (2.2)] (modified form):

$$\int_0^{\infty} y^{w'-1} \sin y\, J_\nu(y)\, H^{m,n}_{p,q}\left[z y^{-2k} \,\middle|\, \begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right] dy$$

$$= 2^{w'-1}\sqrt{\pi}\, H^{m+1,n+1}_{p+1,q+4}\left[2^{-2k}z \,\middle|\, \begin{matrix}(1-(1/2+w'),2k),(a_j,\alpha_j)_{1,p},\\((1+\nu+w')/2,k),(b_j,\beta_j)_{1,q},((w'-\nu)/2,k),((\nu+w')/2,k),((1+w'-\nu)/2,k)\end{matrix}\right] \tag{1.3}$$

where $k > 0$, $|\arg z| < \frac{1}{2} A\pi$, $\mathrm{Re}(w'+\nu) + 2k \min\limits_{1\le j\le n}[\mathrm{Re}\, b_j/\beta_j] > 0$.

The orthogonal property of the Bessel functions [6, p. 291, (6)]:

$$\int_0^{\infty} x^{-1} J_{a+2n+1}(x)\, J_{a+2m+1}(x)\, dx$$

$$= \begin{cases} 0, & \text{if } m \ne n;\\ (4n+2a+2)-1, & \text{if } m = n,\ \mathrm{Re}(a+m+n) > -1.\end{cases} \tag{1.4}$$

The following orthogonal property:

$$\int_0^{\pi} e^{2imx} \cos 2nx\, dx = \begin{cases} 0, & m \ne n\\ \pi/2, & m = n \ne 0\\ \pi, & m = n = 0.\end{cases} \tag{1.5}$$

2. Two Dimensional Exponential – Bessel Partial Differential Equation:

Let us consider

$$\partial u/\partial t = c\, \partial^2 u/\partial t^2 + y^2\, \partial^2 u/\partial y^2 + y\, \partial u/\partial y + y^2 u, \qquad (2.1)$$

where $u \equiv u(x, y, t)$ and $u(x, y, 0) = f(x, y)$.

To solve (2.1), we assume that (2.1) has a solution of the form:

$$u(x, y, t) = e^{4cr^2 t + (\nu + 2s + 1)^2 t}\, X(ix)\, Y(y). \qquad (2.2)$$

The substitution of (2.2) into (2.1) yields:

$$-\, c\, [X'' + 4r^2 X]\, Y + X\, [y^2 Y'' + yY' + \{y^2 - (\nu + 2s + 1)^2\} Y] = 0 \qquad (2.3)$$

We see that $X'' + 4r^2 X = 0$ has a solution $X = e^{2rix}$ and $y^2 Y'' + yY' + \{y^2 - (\nu + 2s + 1)^2\}Y = 0$ is Bessel equation [1, p. 200, (6.25)], with solution $Y = J_{\nu + 2s + 1}(y)$. Therefore the solution of (2.1) is of the form:

$$u(x, y, t) = e^{4cr^2 t + (\nu + 2s + 1)^2 t}\, e^{2rix}\, J_{\nu + 2s + 1}(y). \qquad (2.4)$$

In view of the principal of superposition, the general solution of (2.1) is given by

$$u(x, y, t) = \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} A_{r,s}\, e^{4cr^2 t + (\nu + 2s + 1)^2 t + 2rix}\, J_{\nu + 2s + 1}(y). \qquad (2.5)$$

In (2.5), putting $t = 0$, we get

$$f(x, y) = \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} A_{r,s}\, e^{2rix}\, J_{\nu + 2s + 1}(y). \qquad (2.6)$$

Multiplying both sides of (2.6) by $y^{-1} \cos 2ux\, J_{\nu + 2w + 1}(y)$, integrating with respect to y from 0 to ∞ and with respect to x from 0 to π, then using (1.4) and (1.5), the Fourier Exponential – Bessel coefficients are given by

$$A_{r,s} = (4/\pi)\, (\nu + 2s + 1) \times \int_0^{\pi} \int_0^{\infty} f(x, y)\, y^{-1} \cos 2rx\, J_{\nu + 2w + 1}(y)\, dy\, dx \qquad (2.7)$$

In the view of the theory of double and multiple Fourier series given by Carslaw and Jaeger [3, pp. 180-183], and many other references, such as Erdelyi [4, pp. 64-65] etc., the double series (2.6) is convergent, provided the function f(x, y) is defined in the region $0 < x < p$, $0 < y < \infty$. In brief, the double series (2.6) converges, if the double integral on the right hand side of (2.7) exists.

In the subsequent section, we take f(x, y) as Fox's H-function and present another method to obtain Fourier exponential – Bessel coefficients $A_{r,s}$.

3. Particular Solution Involving Fox's H-Function:

The particular solution to be obtained is

$$u(x, y, t) = 2^{w'+1} \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} e^{4cr^2t + (\nu + 2s + 1)^2t + 2rix} (\nu + 2s + 1) J_{\nu+2s+1}(y).$$

$$\times H^{m+2,\, n+2}_{p+3,\, q+6}\left[2^{-2k}z \,\middle|\, \begin{matrix} (1/2 + w, h), (1-(1/2+w'), 2k), (a_j, \alpha_j)_{1,p}, (w, h) \\ (2u + w, h), ((1+\nu+w')/2, k), (b_j, \beta_j)_{1,q}, (w - 2u, h) \\ ((w'-\nu)/2, k), ((\nu+w')/2, k), ((1+w'-\nu)/2, k) \end{matrix}\right] \qquad (3.1)$$

valid under the conditions of (1.2), (1.3) and (1.4).

Proof: Let

$$f(x, y) = (\sin x/2)^{-2w} y^{w'} \sin y\, H^{m,\, n}_{p,\, q}\left[z (\sin x/2)^{2h} y^{-2k} \,\middle|\, \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix}\right] dx$$

$$= \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} A_{r,s} e^{2rix} J_{\nu+2s+1}(y). \qquad (3.2)$$

Equation (3.2) is valid, since f(x, y) is defined in the region $0 < x < p$, $0 < y < \infty$.

Multiplying both sides of (3.2) by $y^{-1} J_{\nu+2w+1}(y)$ and integrating with respect to y from 0 to ∞, then using (1.3) and (1.4). Now multiplying both sides of the resulting expression by cos 2ux and integrating with respect to x from 0 to π, then using (1.2) and (1.5), we obtain the value of $A_{r,s}$. Substituting the value of $A_{r,s}$ in (2.5), the expansion (3.1) is obtained.

Note 1: The value of $A_{0,s}$ is one-half the value of $A_{r,s}$.
Note 2: If we put t = 0 in (3.1), it reduces to a new two dimensional series expansion for Fox's H-function involving exponential functions and Bessel functions.

Since on specializing the parameters Fox's H-function yields almost all special functions appearing in applied mathematics and physical sciences. Therefore, the result (3.1) presented in this paper is of a general character and hence may encompass several cases of interest.

REFERENCES

1. Andrews, L. C.: Special Functions for Engineers and applied mathematics, Macmillan Publishing Co., New York (1985).
2. Bajpai, S. D.: Fourier Series of Generaliged Hypergeometric Functions, Proc. Camb. Phill. Soc. 65 (1969), 703-707.
3. Carslaw, H. S. and Jaeger, J. C.: Conduction of Heat in Solids (2nd Edt.), Clarendon Press, Oxford, 1986.

4. Erdelyi, A.: Higher Transcendental Functions, Vol. 2, MC Graw-Hill, New York (1953).
5. Fox, C.: The G and H-functions as symmetrical Fourier Kernals, Trans. Amer. Math. Soc. 98 (1961), 395-429.
6. Luke, Y. L.: Integrals of Bessel Functions, Mc Graw-Hill, New York (1962).
7. Taxak, R. L.: Some results involving Fox's H-function and Bessel functions, math. Ed. (Siwan), IV-3, 93-97.

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Some Results on Fixed Points of Integral Function in Menger Spaces

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ABSTRACT : In this paper, we obtain some fixed point theorems using integral type inequality in Menger space employing the property (E.A). Our results improve and generalize several known fixed point theorems existing in the literature.

KEYWORDS : Menger space, Integral Function, Weakly compatible mappings, Property (E.A).

MATHEMATICS SUBJECT CLASSIFICATION (2010) : Primary 47H10, Secondary 54H25.

INTRODUCTION: In the year 1942 Menger [21] introduced the notion of a probabilistic metric space (PM- space) which was, in fact, a generalization of metric space. The idea behind this is to associate a distribution function with a pair of points, say (p,q), denoted by $F_{p,q}(t)$ where $t > 0$ and interpret this function as the probability that distance between p and q is less than t, whereas in the metric space, the distance function is a single positive number. Sehgal [37] initiated the study of fixed points in probabilistic metric spaces. The study of these spaces was expanded rapidly with the pioneering works of Schweizer and Sklar [7]. Jungck [13] introduced the notion of compatible mappings and utilized the same to improve commutativity conditions in common fixed point theorems. This concept has been frequently employed to prove existence theorems on common fixed points. However, the study of common fixed points of non-compatible mappings was initiated by Pant [29]. Recently, Aamri and Moutawakil [1] and Liu et al. [34] respectively defined the property (E.A) and the common property (E.A) and proved interesting common fixed point theorems in metric spaces. Most recently, Kubiaczyk and Sharma [15] adopted the property (E.A) in PM spaces and used it to prove results on common fixed points. Recently, Imdad et al. [26] adopted the common property (E.A) in PM spaces and proved some coincidence and common fixed point results in Menger spaces.

1. PRELIMINARIES : Before going to our main result we require some more definitions and Lemma**,**

Definition 1.1 [8] **:** Let X be a non-empty set and L denote the set of all distribution functions. A probabilistic metric space is an ordered pair (X,F) where $F : X * X \rightarrow L$. we shall denote the distribution function by F (p, q) or $F_{p,q}$; $p, q \in X$ and F(p, q, x) will represent the value of F (p, q) at $x \in R$. the function $F_{p,q}$ is assumed to satisfy the following conditions :

1. $F_{p,q}(t) = 1, \forall t > 0$ if and if $p = q$

2. F p, q (0) = 0 for every p, q ϵ X

3. F p, q (t) = F q, p (t) for every p, q ϵ X

4. If F p, q (t) = 1 and F q, r (s) = 1 it follows that F q r (t +s) =1 ∀ p, q, r ϵ X and t ,s ≥ 0.

In metric space (X ,d) , the metric d induces a mapping F : X * X→L such that F p, q (t) = H(t-d(p, q)) for all p, q ϵ X and t ϵ R , where H is the distribution function defined as

$$H(x) = \begin{cases} 0, \text{if } x \leq 0 \\ 1, \text{if } x > 0 \end{cases}$$

Definition 1.2 [8] **:**A mapping Δ: [0, 1] * [0,1] →[0,1] is called t- norm if the following conditions are satisfied

(1) Δ (a, 1) = a for all a ϵ [0, 1], Δ (0,0) = 0,

(2) Δ (a ,b) = Δ (b, a)

(3) Δ (c, d) ≤ Δ (a, b) for c ≥ a, d ≥ b, and

(4) Δ (Δ (c, d),c) = Δ (a,Δ (b, c)) for all a, b, c ϵ [0,1]

Example 1[8]. The following are the four basic t-norms:

(i) The minimum t-norm: T_M(a, b) = min{a,b}.

(ii) The product t-norm: T_P (a,b) = a.b

(iii) The Lukasiewicz t-norm: T_L(a, b) = max{a + b − 1, 0}.

(iv) The weakest t-norm, the drastic product:

$$T_D(a, b) = \begin{cases} \min\{a,b\}, & if\ \max\{a,b\} = 1 \\ 0, & otherwise \end{cases}$$

In respect of above mentioned t-norms, we have the following ordering:

$T_D < T_L < T_P < T_M$.

Definition 1.3 [21]**:** A Menger probabilistic space is a triplet (X, F, Δ) where (X, F) is a PM-space and Δ is a t- norm with the following condition

$F_{p,r}$ (t +s) ≥ Δ ($F_{p,r}$ (t), $F_{p,r}$ (s)) for all p, q, r ϵ X and t, s ≥ 0.

The above inequality is called Menger's triangle inequality.

Definition 1.4 [28] **:** A sequence {x_n} in (X, F, Δ) is said to be a convergent to a point x ϵ X if for every ε > 0 and λ > 0, there exists an integer N=N (ε ,λ) such that $F_{x_n, x,}$(ε) →1- λ ∀ n ≥ N (ε ,λ).

Definition 1.5 [28] **:** A sequence {x_n} in (X, F, Δ) is said to be a Cauchy sequence if for every ε > 0 and λ >0 , there exists an integer N=N (ε ,λ) such that $F_{x_n,x_m,}$(ε) →1- λ ∀ n, m ≥ N (ε ,λ).

Definition 1.6 [28] **:** A Menger Space (X, F, Δ) with the continuous t- norm is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition 1.7 [24]**:** Let (X, F, Δ) be a Menger PM Space . A pair (f, g) of self mapping on X is said to be weakly commuting if and only if $F_{fgx\,,\,gfx,}$(t) ≥ $F_{fx\,,\,gx,}$(t) for each x ϵ X and t > 0.

Definition 1.8 [31]**:** Let (X, F, Δ) be a Menger PM Space . A pair (f, g) of self mapping on X is said to be compatible if and only if $F_{fgx_n\,,\,gfx_n\,,}$ (t) →1 for all t > 0 whenever {x_n} in X such that fx_n,gx_n → z for some z ϵ X as n → ∞.

Clearly, a weakly commuting pair is compatible but every compatible pair need not be weakly commuting.

Definition 1.10 [19]**:** Let (X, F, Δ) be a Menger PM Space . A pair (f, g) of self mapping on X is said to be non–compatible if and only if there exist at least one sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n =$ z, for some z ϵ X, implies that $\lim_{n\to\infty} F_{fgx_n, gfx_n}(t_0)$ (for some $t_0 > 0$) is either less than 1 or non-existent.

Definition 1.11 [15] **:** Let (X, F, Δ) be a Menger PM Space . A pair (f, g) of self mapping on X is said to satisfy the property (E.A) if there exist a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z, \text{ for some } z \in X.$$

Clearly, a pair of compatible mappings as well as non- Comatible mappings satisfies the property (E.A).

Inspired by Liu et al. [39], Imdad et al. [26] defined the following:

Definition 1.12 [34]: Two pairs (f, g) and (p, q) of self mappings of a Menger PM space (X,F, Δ) are said to satisfy the common property (E.A) if there exist two sequences $\{x_n\}$, $\{y_n\}$ in X and some t in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} px_n = \lim_{n\to\infty} qx_n = z$$

Definition1.13. [24] Two finite families of self mappings $\{A_i\}$ and $\{B_j\}$ are said to be pairwise commuting if:

(i) $A_iA_j = A_jA_i$, i, j ∈ {1, 2...m},

(ii) $B_iB_j = B_jB_i$, i, j ∈ {1, 2...n},

(iii) $A_iB_j = B_jA_i$, i ∈ {1, 2...m}, j ∈ {1, 2...n}.

2. MAIN RESULT:

The following lemma is useful for the proof of succeeding theorems.

Lemma 2.1 [14]**:** Let (X, F, Δ) be a Menger space. If there exists some k ∈ (0, 1) such that for all p, q ∈ X and all x > 0,

$$\int_0^{F_{p,q}(kt)} \emptyset(u)du \geq \int_0^{F_{p,q}(t)} \emptyset(u)du \qquad \text{- - - (2.1.1)}$$

Where ϕ : [0, ∞) → [0, ∞) is a non-negative summable Lebesque integrable function such that

$$\int_\varepsilon^1 \emptyset(u)du > 0 \text{ for each } \varepsilon \in [0,1) \text{ then } p = q.$$

Proof. From (2.1.1)

$$\int_0^{F_{p,q}(t)} \emptyset(u)du \geq \int_0^{F_{p,q}(k^{-1}t)} \emptyset(u)du$$

one can inductively write (for m ∈ N)

$$\int_0^{F_{p,q}(t)} \emptyset(u)du \geq \int_0^{F_{p,q}(k^{-1}t)} \emptyset(u)du \geq - - - \geq \int_0^{F_{p,q}(k^{-m}t)} \emptyset(u)du$$

$$\geq \int_0^1 \emptyset(u)du \quad \text{as } m \to \infty.$$

Therefore

$$\int_0^{F_{p,q}(t)} \emptyset(u)du - \int_0^1 \emptyset(u)du \geq 0$$

And hence,

$$\int_0^{F_{p,q}(t)} \emptyset(u)du \left(\int_0^{F_{p,q}(t)} \emptyset(u)du - \int_0^1 \emptyset(u)du\right) \geq 0$$

Or,

$$\int_{F_{p,q}(t)}^1 \emptyset(u)du \leq 0.$$

which amounts to say that Fp,q(t) ≥ 1 for all t ≥ 0. Thus, we get p = q.

Remark : By setting φ(t) = 1 (for each t ≥ 0) in (2.1.1) of Lemma 2.1, we have

$$\int_0^{F_{p,q}(kt)} \emptyset(u)du = F_{p,q}(kt) \geq F_{p,q}(t) = \int_0^{F_{p,q}(t)} \emptyset(u)du,$$

which shows that Lemma 1 is a generalization of the Lemma 2 (contained in [34]).

In what follows, Δ is a continuous t-norm (in the product topology).

Lemma 2.2 : Let (X.F, Δ) be a complete Menger Space and let f, g, p and q be self mapping of X
satisfying the conditions :

(i) pairs {p, f} and {q, g} satisfies the property E.A.

(ii) B(y_n) converges for every sequence {y_n} in X whenever T(y_n) converges,

(iii) for any x,y ∈ X and for all t > 0,

$$\int_0^{F_{px,qy}(kt)} \emptyset(u)du \geq \int_0^{m(x,y)} \emptyset(u)du \qquad \text{- - - (2.2.1)}$$

Where ϕ : [0, ∞) → [0, ∞) is a non-negative summable Lebesque integral function such that

$\int_c^1 \emptyset(u)du > 0$ for each u ∈ [0,1), where 0 < k < 1 and

m(x,y) =

$$\min\{F_{fx,gy}(t), F_{fx,px}(t), F_{gy,qy}(t), F_{fx,qy}(t), F_{gy,px}(t), \frac{F_{fx,gy}(t).\,F_{gy,qy}(t)}{F_{fx,qy}(t)}, \frac{F_{fx,gy}(t).\,F_{fx,,px}(t)}{F_{gy,px}(t)}$$

(iv) p(X) ⊂ g(X) (or q(X) ⊂ f(X)).

Then the pair (p,f) and (q,g) share the common property (E.A.).

Proof : Suppose that the pair (p,f) enjoys the property (E.A.)**,**then there exist a sequence {x_n}in X such that

$$\lim_{n\to\infty} px_n = \lim_{n\to\infty} fx_n = u, \text{ for some } u \in X.$$

Since p(X) ⊂ g(X), for each x_n there exists y_n ∈ X.such that $px_n = gy_n$, and hence

$$\lim_{n\to\infty} gy_n = \lim_{n\to\infty} px_n = u$$

Thus in all, we have $px_n \to u$, $fx_n \to u$ and $gy_n \to u$. Now we assert that $qy_n \to u$.

To accomplish this, using (2.2.1) with x = x_n and y = y_n, we get

$$\int_0^{F_{px_n,qy_n}(kt)} \emptyset(u)du \geq \int_0^{m(x_n,y_n)} \emptyset(u)du$$

Where, $m(x_n, y_n) =$

$$\min\{F_{fx_n,gy_n}(t),\ F_{fx_n,px_n}(t),\ F_{gy_n,qy_n}(t),\ F_{fx_n,qy_n}(t),\ F_{gy_{n,},px_n}(t),$$
$$\frac{F_{fx_n,qy_n}(t).\,F_{gy_n,qy_n}(t)}{F_{fx_n,qy_n}(t)},\ \frac{F_{fx_n,gy_n}(t).\,F_{fx_n,px_n}(t)}{F_{gy_n,px_n}(t)}\}$$

Let, $\lim_{n\to\infty} q(y_n)$ = v

Also, let t > 0 be such that $F_{u,v}(.)$ is continuous in t and kt.

Then, on making n → ∞ in the above inequality, we get

$$\int_0^{F_{u,v}(kt)} \emptyset(u)du \geq$$

$$\int_0^{\min\{F_{u,u}(t), F_{fu,u}(t), F_{u,v}(t), F_{u,v}(t), F_{u,u}(t), \frac{F_{u,,u}(t).\, F_{u,,v}(t)}{F_{u,,v}(t)}, \frac{F_{u,u}(t).\, F_{u,u}(t)}{F_{u,u}(t)},} \emptyset(u)du$$

Or, $\int_0^{F_{u,v}(kt)} \emptyset(u)du \geq \int_0^{F_{u,v}(t)} \emptyset(u)du$

This, implies that v = u (in view of Lemma 2.1) which shows that the pair (p,f) and (q,g) share the common property (E.A).

Theorem 2.3 : Let f, g, p and q be self mappings of a Menger space (X, F, Δ) which satisfy the inequality (2.2.1) together with the conditions :

(i) the pairs (p, f) and (q, g) share the common property (E.A),

(ii) f(X) and g(X) are closed subsets of X.

Then the pairs (p, f) and (q, g) have a point of coincidence each. Moreover, f, g, p and q have a unique common fixed point provided both the pairs (p, f) and (q, g) are weakly compatible.

Proof. Since the pairs (p, f) and (q, g) share the common property (E.A), there exist two sequences $\{x_n\}$
and $\{y_n\}$ in X such that

$\lim_{n\to\infty} px_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} qy_n =$ u, for some u ∈ X.

Since f(X) is a closed subset of X, hence $\lim_{n\to\infty} fx_n$ = u ∈ f(X).

Therefore, there exists a point z ∈ X such that fz = u.

Now, we assert that pz = fz.

To prove this, on using (2.2.1) with x = z, y = y_n, we get

$$\int_0^{F_{pz,\,qy_n}(kt)} \emptyset(u)du \geq$$

$$\int_0^{\min\{F_{fz,qy_n}(t), F_{fz,pz}(t), F_{gy_n,\,qy_n}(t), F_{fz,\,qy_n}(t), F_{gy_n,pz}(t), \frac{F_{fz,qy_n}(t).\, F_{gy_n,qy_n}(t)}{F_{fz,qy_n}(t)}, \frac{F_{fz,gy_n}(t).\, F_{fz,,pz}(t)}{F_{gy_n,pz}(t)}\}} \emptyset(u)du$$

On taking n ⟶ ∞, reduces to

$$\int_0^{F_{pz,u}(kt)} \emptyset(u)du \geq$$

$$\int_0^{min\left\{F_{u,u}(t), F_{u,,pz}(t), F_{u,u}(t), F_{u,,u}(t), F_{u,,pz}(t), \frac{F_{u,,u}(t).\, F_{u,u}(t)}{F_{u,u}(t)}, \frac{F_{u,u}(t).\, F_{u,,pz}(t)}{F_{u,,pz}(t)}\right\}} \emptyset(u)du$$

$$\int_0^{F_{pz,u}(kt)} \emptyset(u)du \geq \int_0^{F_{pz,u}(t)} \emptyset(u)du$$

Now on appealing Lemma 2.1, we get pz = u and hence pz = fz. Therefore, z is a coincidence point
of the pair (p, f).

Since g(X) is a closed subset of X, therefore $\lim_{n\to\infty} gy_n$ = u ∈ g(X) and hence we can find a point
w∈ X such that gw = u

Now we show that qw = gw.

To accomplish this, on using (2.2.1) with x = x_n, y = w, we have

$$\int_0^{F_{px_n,qw}(kt)} \emptyset(u)du \geq \int_0^{m(x_n,w)} \emptyset(u)du$$

Where, $m(x_n,w) = \min\{F_{fx_n,qw}(t),\ F_{fx_n,px_n}(t),\ F_{gw,qw}(t),\ F_{fx_n,qw}(t),\ F_{gw,px_n}(t),$
$\frac{F_{fx_n,qw}(t).\,F_{gw,qw}(t)}{F_{fx_n,qw}(t)}, \frac{F_{fx_n,gw}(t).\,F_{fx_n,px_n}(t)}{F_{gw,px_n}(t)}\}$

Which on making n → ∞, reduces to

$$\int_0^{F_{u,qw}(kt)} \emptyset(u)du \geq$$

$$\int_0^{\min\{F_{u,u}(t),\ F_{u,,u}(t), F_{u,,qw}(t), F_{u,,qw}(t), F_{u,u}(t),\ \frac{F_{u,u}(t).\,F_{u,qw}(t)}{F_{u,qw}(t)},\ \frac{F_{u,u}(t).\,F_{u,u}(t)}{F_{gw,u}(t)}\}} \emptyset(u)du$$

$$\int_0^{F_{u,qw}(kt)} \emptyset(u)du \geq \int_0^{F_{u,qw}(t)} \emptyset(u)du$$

on employing Lemma 2.1, we get qw = u and gw = qw

Therefore, w is a coincidence point of the pair (q, g).

Since the pair (p, f) is weakly compatible and pz = fz, therefore pu = pfz = fpz = fu.

Again, on using (2.2.1) with x = u, y = w, we have

$$\int_0^{F_{pu,qw}(kt)} \emptyset(u)du \geq \int_0^{m(u,w)} \emptyset(u)du$$

Where, $m(u,w) =$
$\min\{F_{fu,qw}(t),\ F_{fu,pu}(t),\ F_{gw,qw}(t),\ F_{fu,qw}(t), F_{gw,pu}(t),\ \frac{F_{fu,gw}(t).\,F_{gw,qw}(t)}{F_{fu,qw}(t)},$
$\frac{F_{fu,gw}(t).\,F_{fu,pu}(t)}{F_{gw,pu}(t)}\}$

Or,

$$\int_0^{F_{pu,u}(kt)} \emptyset(u)du \geq$$

$$\int_0^{\min\{F_{fu,u}(t),\ F_{pu,pu}(t), F_{u,,u}(t), F_{pu,u}(t), F_{u,pu}(t),\ \frac{F_{pu,u}(t).\,F_{u,,u}(t)}{F_{pu,u}(t)},\ \frac{F_{pu,u}(t).\,F_{pu,,pu}(t)}{F_{u,,pu}(t)}\}} \emptyset(u)du$$

Or,

$$\int_0^{F_{pu,u}(kt)} \emptyset(u)du \geq \int_0^{F_{pu,u}(t)} \emptyset(u)du$$

On employing Lemma 2.1, we have pu = fu = u, which shows that u is a common fixed point of
the pair (p, f).

Also the pair (q, g) is weakly compatible and qw = gw, hence

qu = qgw = gqw = gu.

Next, we show that u is a common fixed point of the pair (q, g). In order to accomplish this, using (2.2.1) with x = z, y = u, we get

$$\int_0^{F_{pz,qu}(kt)} \emptyset(u)du \geq$$

$$\int_0^{\min\{F_{fz,gu}(t),\ F_{fz,pz}(t), F_{gu,qu}(t), F_{fz,qu}(t), F_{gu,pz}(t),\ \frac{F_{fz,gu}(t).\,F_{gu,qu}(t)}{F_{fz,qu}(t)},\ \frac{F_{fz,gu}(t).\,F_{fz,,pz}(t)}{F_{gu,pz}(t)}\}} \emptyset(u)du$$

Or,

$$\int_0^{F_{u,qu}(kt)} \emptyset(u)du \geq$$

$$\int_0^{\min\{F_{u,gu}(t),\ F_{u,,u}(t), F_{qu,qu}(t), F_{u,,qu}(t), F_{qu,u}(t),\ \frac{F_{u,qu}(t).\,F_{qu,qu}(t)}{F_{u,qu}(t)},\ \frac{F_{u,qu}(t).\,F_{u,,u}(t)}{F_{qu,u}(t)}\}} \emptyset(u)du$$

Or,

$$\int_0^{F_{u,qu}(kt)} \emptyset(u)du \geq \int_0^{F_{u,qu}(t)} \emptyset(u)du$$

Using Lemma 2.1, we have qu = u which shows that u is a common fixed point of the pair (q, g). Hence

u is a common fixed point of both the pairs (p, f) and (q, g). Uniqueness of common fixed point is an

easy consequence of the inequality (2.2.1). This completes the proof.

REFERENCES

[1]. A. Aamri and D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, J.Math.Anal.Appl.270(2002),181-188.

[2]. A. Aliouche, A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type, J. Math. Anal. Appl., 322(2)(2006), 796-802.

[3]. A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int. J. Math. Math. Sci., 29(9)(2002), 531-536.

[4]. A. Djoudi and A. Aliouche, Common fixed point theorems of Gregus type for weakly compatible Mappings satisfying contractive conditions of integral type, J. Math. Anal. Appl., 329 (1) (2007), 31-45.

[5]. A. Razani and M. Shirdaryazdi, A common fixed point theorem of compatible maps in Menger space,Chaos, Solitions and Fractals, 32 (2007), 26-34.

[6]. B. E. Rhoades, A comparison of various definitions of contractive mappings, Trans. Amer. Math. Soc., 226 (1977), 257-290.

[7]. B. E. Rhoades, Two fixed point theorems for mapping satisfyng a general contractive condition of integral type, Int. J. Math. Math. Sci., 63 (2003), 4007-4013.

[8]. B. Schweizer and A. Sklar, Probabilistic metric spaces, Elsevier, North Holand, New York, 1983.

[9]. B. Singh and S. Jain, A fixed point theorem in Menger spaces through weak compatibility, J. Math. Anal. Appl., 301 (2005), 439-448.

[10]. D. Mihet, A generalization of a contraction principle in probabilistic metric spaces (II), Int. J. Math. Math. Sci 5 (2005), 729-736.

[11]. D. Mihet, Fixed point theorems in fuzzy metric spaces using property E.A., Nonlinear Anal., 73 (2010), 2184-2188.

[12]. Dr. kamal Wadhwa, dr. ramakant bhardwaj and Jyoti panthi, Common fixed point theorems of integral type in Menger PM Spaces, Network and Complex Systems, vol.3, No.6(2013), 10-16.

[13]. G. Jungck, Common fixed points for non continuous nonself maps on nonmetric spaces. Far East J. Math. Sci., 4 (2) (1996), 199-215.
[14]. I. Altun, M. Tanveer and M.Imdad, Common fixed point theorems of integral type in Menger PM spaces, Journal of Nonlinear Analysis and Optimization, vol.(3) No.1(2012),55-56.
[15]. I. Kubiaczyk and S. Sharma, Some common fixed point theorems in Menger space under strict Contractive conditions, Southeast Asian Bull. Math., 32 (2008), 117-124.
[16]. I. N. Beg, Shazad and M. Iqbal, Fixed points theorems and best approximation in convex metric spaces, J. Approx. Theory Appl. 8(4)(1992), 97-105.
[17]. J. Ali, M. Imdad and D. Bahuguna, Common fixed point theorems in Menger spaces with common property (E.A), Comp. Math. Appl., 60(2010), 3152-3159.
[18]. J. Ali, M. Imdad, D. Mihe¸t, and M. Tanveer, Common fixed points of strict contractions in Menger spaces, Acta Math. Hung., 132(4)(2011), 367-386.
[19]. J.X. Fang and Y. Gao, Common fixed point theorems under strict contractive conditions in Menger spaces, Nonlinear Analysis, 70 (2009). 184-193.
[20]. K. Menger, Probabilistic geometry, Proc. Nat. Acad. Sci. USA, 37 (1951), 226-229.
[21]. K. Menger, Statistical metrics, Proc. Nat. Acad. Sci. USA, 28 (1942), 535-537.
[22]. Lj. B. Ciri´c, A generalization of Banach’s contraction principle, Proc. Amer. Math. Soc., 45 (1974), 267- 273.
[23]. M. Imdad and J. Ali, Jungck’s common fixed point theorem and E.A property, Acta Math. Cinica, 2(2008), 87-94.
[24]. M. Imdad, Javid Ali and M. Tanveer, Coincidence and common fixed point theorems for nonlinear contractions in Menger PM spaces, Chaos, Solitons & Fractals, 42 (2009), 3121-3129.
[25]. M. Imdad, M. Tanveer and M. Hasan, Erratum to “Some common fixed point theorems in Menger PM Spaces”, Fixed Point Theory Appl., 2011, 2011:28.
[26]. M. Imdad, M. Tanveer and M. Hasan, Some commn fixed point theorems in Menger PM spaces, Fixed Point Theory Appl., Vol. 2010, 14 pages.
[27]. R. Chugh and S. Rathi, Weakly compatible maps in probabilistic metric spaces, J. Indian Math. Soc., 72 (2005), 131-140.
[28]. O. Hadzic and E. Pap, Fixed point theory in probabilistic metric spaces, Kluwer Academic Publishers, Dordrecht, 2001.
[29]. R.P. Pant, Common fixed points of noncommuting mappings, J. Math. Anal. Appl., 188 (1994), 436-440.
[30]. S. A. Naimpally, K.L Singh and J.H.M. Whitefield, Fixed points and non-expansive retracts in locally convex spaces, Fund. Math., 120 (1984), 63-75.

[31]. S.N. Mishra, Common fixed points of compatible mappings in PMspaces, Math. Japon., 36 (1991), 283- 289.
[32]. T. Suzuki, Meir Keeler contractions of integral type are still MeirKeeler contractions, Internat. J. Math. Math. Sci., (2007), Article ID 39281, 6 pages.
[33]. V.M. Sehgal and A.T. Bharucha-Reid, Fixed point of contraction mappings on probabilistic metric spaces, Math. Systems Theory, 6 (1972), 97-102.
[34]. Y. Liu, Jun Wu and Z. Li, Common fixed points of single-valued and multivalued maps, Int. J. Math. Math. Sci., 19 (2005), 3045-3055.

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